

Kaplansky's zero divisor and unit conjectures on elements with supports of size 3

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Abstract. Kaplansky's zero divisor conjecture (unit conjecture, respectively) states that for a torsion-free group G and a field \mathbb{F} , the group ring $\mathbb{F}[G]$ has no zero divisors (has no unit with support of size greater than 1). In this paper, we study possible zero divisors and units in $\mathbb{F}[G]$ whose supports have size 3. For any field \mathbb{F} and all torsion-free groups G , we prove that if $\alpha\beta = 0$ for some non-zero $\alpha, \beta \in \mathbb{F}[G]$ such that $|supp(\alpha)| = 3$, then $|supp(\beta)| \geq 10$. If $\mathbb{F} = \mathbb{F}_2$ is the field with 2 elements, the latter result can be improved so that $|supp(\beta)| \geq 20$. This improves a result in [J. Group Theory, 16 (2013), no. 5, 667-693]. Concerning the unit conjecture, we prove that if $\alpha\beta = 1$ for some $\alpha, \beta \in \mathbb{F}[G]$ such that $|supp(\alpha)| = 3$, then $|supp(\beta)| \geq 9$. The latter improves a part of a result in [Exp. Math., 24 (2015), 326-338] to arbitrary fields.

1. Introduction and Results

Let R be a ring. A non-zero element α of R is called a zero divisor if $\alpha\beta = 0$ or $\beta\alpha = 0$ for some non-zero element $\beta \in R$. Let G be a group. Denote by $R[G]$ the group ring of G over R . If R contains a zero divisor, then clearly so does $R[G]$. Also, if G contains a non-identity torsion element x of finite order n , then $R[G]$ contains zero divisors $\alpha = 1 - x$ and $\beta = 1 + x + \cdots + x^{n-1}$, since $\alpha\beta = 0$. Around 1950, Irving Kaplansky conjectured that existence of a zero divisor in a group ring depends only on the existence of such elements in the ring or non-trivial torsions in the group by stating one of the most challenging problems in the field of group rings [11].

Conjecture 1.1 (Kaplansky's zero divisor conjecture). *Let \mathbb{F} be a field and G be a torsion-free group. Then $\mathbb{F}[G]$ does not contain a zero divisor.*

Another famous problem, namely the unit conjecture, also proposed by Kaplansky [11], states that:

Conjecture 1.2 (Kaplansky's unit conjecture). *Let \mathbb{F} be a field and G be a torsion-free group. Then $\mathbb{F}[G]$ has no non-trivial units (i.e., non-zero scalar multiples of group elements).*

It can be shown that the zero divisor conjecture is true if the unit conjecture has an affirmative solution (see Lemma 13.1.2 in [16]).

Over the years, some partial results have been obtained on Conjecture 1.1 and it has been confirmed for special classes of groups which are torsion free. One of the first known special families which satisfy Conjecture 1.1 are unique product groups [16, Chapter 13], in particular ordered groups. Furthermore,

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by the fact that Conjecture 1.1 is known to hold valid for amalgamated free products when the group ring of the subgroup over which the amalgam is formed satisfies the Ore condition [15], it is proved by Formanek [8] that supersolvable groups are another families which satisfy Conjecture 1.1. Another result, concerning large major sorts of groups for which Conjecture 1.1 holds in the affirmative, is obtained for elementary amenable groups [13]. The latter result covers the cases in which the group is polycyclic-by-finite, which was firstly studied in [2] and [6], and then extended in [20]. Some other affirmative results are obtained on congruence subgroups in [14] and [7], and certain hyperbolic groups [3]. Nevertheless, Conjecture 1.1 has not been confirmed for any fixed field and it seems that confirming the conjecture even for the smallest finite field \mathbb{F}_2 with two elements is still out of reach.

The support of an element $\alpha = \sum_{x \in G} a_x x$ of $R[G]$, denoted by $\text{supp}(\alpha)$, is the set $\{x \in G \mid a_x \neq 0\}$. For any division ring \mathbb{K} and all torsion-free group G , it is known that $\mathbb{K}[G]$ does not contain a zero divisor whose support is of size at most 2 (see [4, Proposition 2.6] and also [19, Theorem 2.1] when \mathbb{K} is assumed to be a field), but it is not known a similar result for group ring elements with the support of size 3. By describing a combinatorial structure, named matched rectangles, Schweitzer [19] showed that if $\alpha\beta = 0$ for $\alpha, \beta \in \mathbb{F}_2[G] \setminus \{0\}$ when $|\text{supp}(\alpha)| = 3$, then $|\text{supp}(\beta)| > 6$. Also, with a computer-assisted approach, he showed that if $|\text{supp}(\alpha)| = 3$, then $|\text{supp}(\beta)| > 16$.

Let G be an arbitrary torsion-free group and let $\alpha \in \mathbb{F}[G]$ be a possible zero divisor such that $|\text{supp}(\alpha)| = 3$ and $\alpha\beta = 0$ for some non-zero $\beta \in \mathbb{F}[G]$. In this paper, we study the minimum possible size of the support of such an element β . Let β have minimum possible support size and $\mathbb{F} = \mathbb{F}_2$. In [19, Definition 4.1] a graph is associated to the non-degenerate $3 \times |\text{supp}(\beta)|$ matched rectangle corresponding to α and β and it is proved in [19, Theorem 4.2] that the graph is a simple cubic one without triangles. We call the graph Kaplansky graph of (α, β) over \mathbb{F}_2 and it is denoted by $K_{\mathbb{F}_2}(\alpha, \beta)$. We extend such definition to the case that \mathbb{F} is an arbitrary field and the corresponding Kaplansky graph is denoted by $K_{\mathbb{F}}(\alpha, \beta)$. So, any Kaplansky graph is derived from a possible zero divisor with support of size 3 in the group algebra of a torsion-free group over the field \mathbb{F} . In fact $K_{\mathbb{F}}(\alpha, \beta)$ is the induced subgraph on the set $\text{supp}(\beta)$ of the Cayley graph $\text{Cay}(G, S)$ ¹, where $S = \{h^{-1}h' \mid h, h' \in \text{supp}(\alpha), h \neq h'\}$. Here we study forbidden subgraphs of Kaplansky graphs. Our main results on Conjecture 1.1 are the followings.

Theorem 1.3. *None of the graphs in Figure 1 can be isomorphic to a subgraph of any Kaplansky graph over any field \mathbb{F} .*

Theorem 1.4. *Let α and β be non-zero elements of the group algebra of any torsion-free group over an arbitrary field. If $|\text{supp}(\alpha)| = 3$ and $\alpha\beta = 0$ then $|\text{supp}(\beta)| \geq 10$.*

Theorem 1.5. *None of the graphs in Table 1 can be isomorphic to a subgraph of any Kaplansky graph over \mathbb{F}_2 .*

In Appendix 8 some details of our computations needed in the proof of Theorem 1.5 are given for the reader's convenience.

The following result improves a result in [19].

¹By a Cayley graph $\text{Cay}(G, S)$ for a group G and a subset S of G with $1 \notin S = S^{-1}$, is the graph whose vertex set is G and two vertices g_1, g_2 are adjacent if $g_1 g_2^{-1} \in S$.

Theorem 1.6. *Let α and β be non-zero elements of the group algebra of any torsion-free group over the field with two elements. If $|supp(\alpha)| = 3$ and $\alpha\beta = 0$ then $|supp(\beta)| \geq 20$.*

The best known result on Conjecture 1.2, which has the purely group-theoretic approach, is concerned with unique product groups [16, 17]. The latter result covers ordered groups, in particular torsion-free nilpotent groups. Nevertheless, it is still unknown whether or not Conjecture 1.2 do hold true for supersolvable torsion-free groups. Dykema et al. [5] have shown that there exist no $\gamma, \delta \in \mathbb{F}_2[G]$ such that $\gamma\delta = 1$, where $|supp(\gamma)| = 3$ and $|supp(\delta)| \leq 11$. Concerning Conjecture 1.2, we prove the following result which improves a part of the result in [5] to arbitrary fields.

Theorem 1.7. *Let γ and δ be elements of the group algebra of any torsion-free group over an arbitrary field. If $|supp(\gamma)| = 3$ and $\gamma\delta = 1$ then $|supp(\delta)| \geq 9$.*

It is known that $\mathbb{F}[G]$ contains a zero divisor if and only if it contains a non-zero element whose square is zero (see [16]). Using the latter fact, it is mentioned in [19, p. 691] that it is sufficient to check Conjecture 1.1 only for the case that $|supp(\alpha)| = |supp(\beta)|$, but in the construction that, given a zero divisor produces an element of square zero, it is not clear how the length changes. We clarify the latter by the following.

Proposition 1.8. *If $\mathbb{F}[G]$ has no non-zero element α with $|supp(\alpha)| \leq k$ such that $\alpha^2 = 0$, then there exist no non-zero elements $\alpha_1, \alpha_2 \in \mathbb{F}[G]$ such that $\alpha_1\alpha_2 = 0$ and $|supp(\alpha_1)||supp(\alpha_2)| \leq k$.*

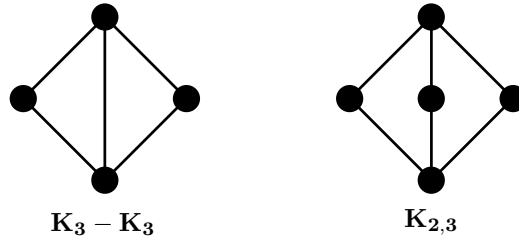
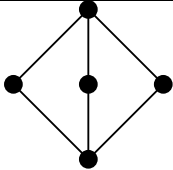
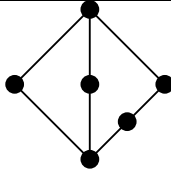
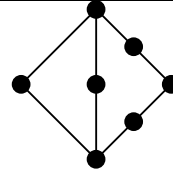
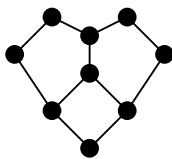
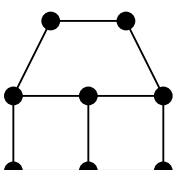
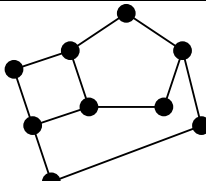
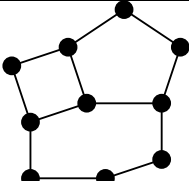
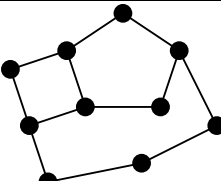


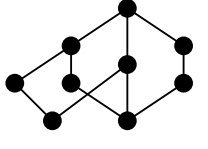
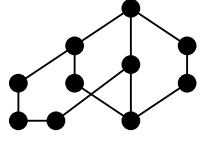
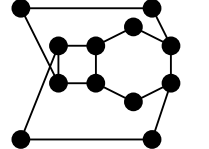
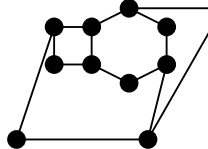
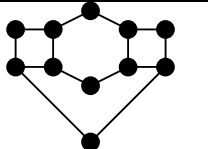
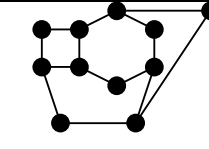
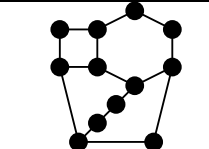
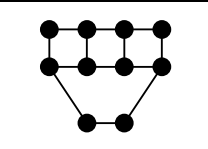
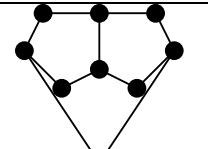
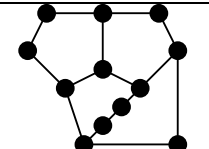
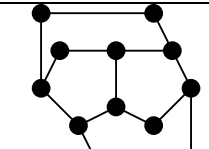
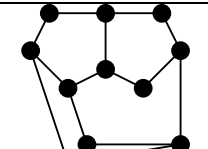
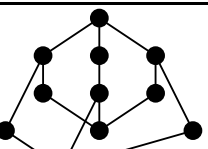
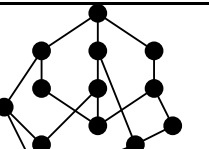
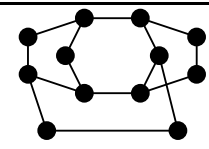
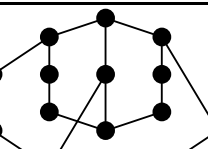
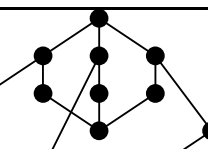
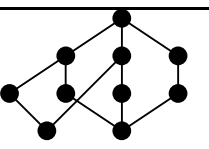
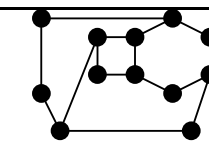
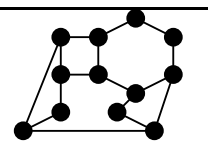
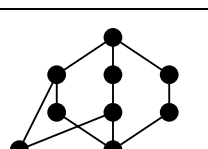
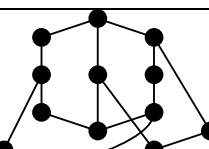
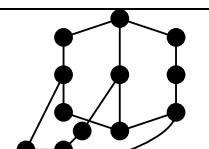
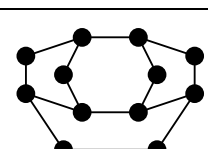
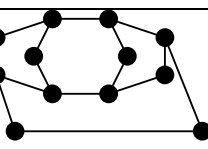
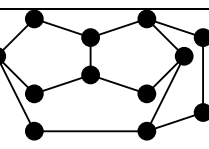
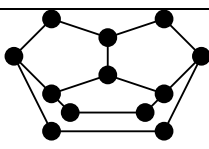
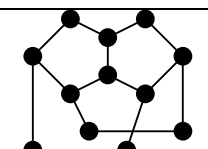
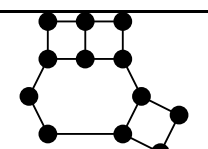
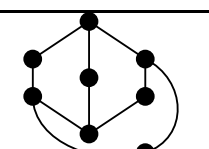
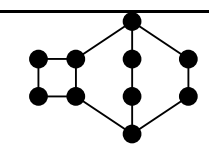
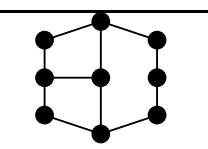
FIGURE 1. Two forbidden subgraphs of the Kaplansky graph over \mathbb{F}

Table 1: Forbidden subgraphs of Kaplansky graphs over \mathbb{F}_2

			
1) $K_{2,3}$	2) $C_4 - C_5$	3) $C_4 - C_6$	4) $C_4 - C_5(-C_5-)$
			
5) $C_4 - C_5(-C_4-)$	6) $C_4 - C_5(-C_6--)$	7) $C_4 - C_5(-C_6-)$	8) $C_4 - C_5(-C_7--)$

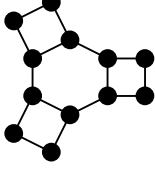
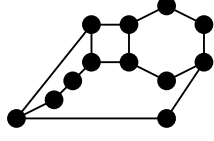
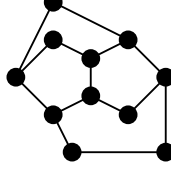
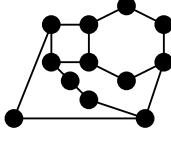
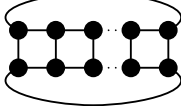
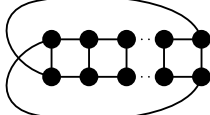
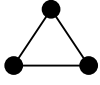
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9) $C_5 - C_5(-C_5)$	10) $C_5 - C_5(-C_6)$	11) $C_4 - C_6(-C_7-)(C_7-1)$	12) $C_4 - C_6(-C_7-)(-C_5-)$
			
13) $C_4 - C_6(-C_6-)(-C_4-)$	14) $C_4 - C_6(-C_6-)(-C_5-)$	15) $C_4 - C_6(-C_6-)(C_6-)$	16) $C_4 - C_6(-C_6-)(-C_4)$
			
17) $C_5 - C_5(-C_6-)$	18) $C_5 - C_5(-C_6-)(C_6-)$	19) $C_5 - C_5(-C_6-)(-C_6-1)$	20) $C_5 - C_5(-C_6-)(-C_5-)$
			
21) $C_6 - C_6(C_6- C_6)$	22) $C_6 - C_6(C_6)(C_6)(C_6)$	23) $C_5(-C_6-)C_5(-C_6-)$	24) $C_6 - C_6(C_6- C_6)$
			
25) $C_6 - C_6(C_6- C_6)$	26) $C_6 - C_6(-C_5-)$	27) $C_4 - C_6(-C_7-)(-C_6-)$	28) $C_4 - C_6(-C_7-)(C_4)(C_4)$
			
29) $C_6 - C_6(-C_5-)$	30) $C_6 - C_6(-C_5-)(-C_5-)$	31) $C_6 - C_6(-C_5-)(C_6-)$	32) $C_5(-C_6-)C_5(C_6)$
			
33) $C_5(-C_6-)C_5(C_7)$	34) $C_5 - C_5(-C_7-)(-C_5-)$	35) $C_5 - C_5(-C_7-)(-C_5-)$	36) $C_5 - C_5(-C_6-)(-C_6-2)$
			
37) $C_4 - C_4(-C_7-)(C_4)$	38) $C_5 - C_5(-C_5-)$	39) $C_6 - C_6(-C_4)$	40) $C_6 - C_6(C_4)$

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41) $C_4 - C_6(-C_4)(-C_4)$	42) $C_4 - C_6(- - C_7 - -)(-C_5 -)$	43) $C_5 - C_5(-C_6 - -)(- - C_5 -)$	44) $C_4 - C_6(- - C_7 - -)(C_7 - 2)$
			
45) L_n	46) M_n	47) K_3	

2. Kaplansky graphs over \mathbb{F} and some of their properties

Throughout this paper let G be a torsion-free group and $\alpha = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3 \in \mathbb{F}[G]$ such that $|supp(\alpha)| = 3$. Suppose further that $\alpha\beta = 0$ for some non-zero $\beta \in \mathbb{F}[G]$ and assume that $n := |supp(\beta)|$ is minimum with respect to the latter property and $\beta = \beta_1 g_1 + \beta_2 g_2 + \dots + \beta_n g_n$. So, $n \geq 3$ [19].

Lemma 2.1 (See also Lemma 2.1 of [4]). $\langle h_i^{-1} supp(\alpha) \rangle = \langle supp(\beta) g_j^{-1} \rangle$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \dots, n\}$.

Proof. Let $i \in \{1, 2, 3\}$, $j \in \{1, \dots, n\}$, $H = \langle h_i^{-1} supp(\alpha) \rangle$, $K = \langle supp(\beta) g_j^{-1} \rangle$ and $\{t_1, t_2, \dots, t_k\}$ be a set of right coset representatives of H in G such that if $k_{j'} \in supp(\beta) g_j^{-1}$, then $k_{j'} \in H t_{i'}$ for some $i' \in \{1, 2, \dots, k\}$. Suppose that $k > 1$. Since $\alpha\beta = 0$ and $H t_{l_1} \cap H t_{l_2} = \emptyset$ for all distinct $l_1, l_2 \in \{1, 2, \dots, k\}$, $(h_i^{-1} \alpha)(a_1 h'_1 + a_2 h'_2 + \dots + a_l h'_l) t_l = 0$ for some $l \in \{1, 2, \dots, k\}$, $h'_1, h'_2, \dots, h'_l \in H$ and $\{a_1, a_2, \dots, a_l\} \subseteq supp(\beta)$, where $i_l < n$, that is a contradiction with the minimality of n because $\alpha(a_1 h'_1 + a_2 h'_2 + \dots + a_l h'_l) = 0$. So $k = 1$, $\alpha(\beta_1 h'_1 t_1 + \beta_2 h'_2 t_1 + \dots + \beta_n h'_n t_1) = 0$ and $supp(\beta) g_j^{-1} = \{h'_1 t_1, h'_2 t_1, \dots, h'_n t_1\}$ where $h'_1, h'_2, \dots, h'_n \in H$. Since $1 \in supp(\beta) g_j^{-1} \subseteq H t_1$, $1 = h t_1$ for some $h \in H$ and so $t_1 \in H$. Therefore, $supp(\beta) g_j^{-1} \subseteq H$ and so $K \leq H$.

Since $\alpha\beta = 0$, $h_i g_j = h_{i'} g_{j'}$ for some $(i', j') \in A = \{1, 2, 3\} \times \{1, 2, \dots, n\}$ such that $i \neq i'$ and $j \neq j'$. So, $h_i^{-1} h_{i'} = g_j g_{j'}^{-1} = (g_{j'} g_j^{-1})^{-1} \in K$. Furthermore, $h_i^{-1} h_i = 1 = g_j g_j^{-1} \in K$. Therefore, $h_i^{-1} h_i, h_i^{-1} h_{i'} \in K$ where $supp(\alpha) = \{h_i, h_{i'}, h_{i''}\}$. Similarly, $h_{i''} g_j = h_s g_t$ for some $(s, t) \in A$ such that $s \neq i''$ and $t \neq j$. Since $h_s \in supp(\alpha) \setminus \{h_{i''}\}$, $h_s = h_i$ or $h_s = h_{i'}$. If $h_s = h_i$, then $h_i^{-1} h_{i''} = g_t g_j^{-1} \in K$. If $h_s = h_{i'}$, then $h_{i'}^{-1} h_{i''} = g_t g_j^{-1} \in K$. So, $h_i^{-1} h_{i''} \in K$ because $h_{i'}^{-1} h_{i''}, h_i^{-1} h_{i'} \in K$ and $h_i^{-1} h_{i''} = h_i^{-1} h_{i'} h_{i'}^{-1} h_{i''}$. Therefore, $h_i^{-1} supp(\alpha) \subseteq K$ which implies that $H = K$ because $K \leq H$. \square

Remark 2.2. By Lemma 2.1, it can be supposed that $G = \langle h_i^{-1} supp(\alpha) \rangle$ for all $i \in \{1, 2, 3\}$. Since $1 \in h_i^{-1} supp(\alpha)$, without loss of generality we may assume that $supp(\alpha) = \{1, h_2, h_3\}$ and $G = \langle supp(\alpha) \rangle$.

The size of $S = \{h^{-1} h' \mid h, h' \in supp(\alpha), h \neq h'\}$ is at most 6 and we prove that S should have its largest possible size.

Lemma 2.3. $|S| = 6$.

Proof. Suppose, for a contradiction, that $|S| < 6$. Then $h_i^{-1}h_j = h_{i'}^{-1}h_{j'}$ for some $(i, j) \neq (i', j')$ and $(i, i') \neq (j, j')$. It follows that $(i', j') = (j, i)$ or (j, k) or (k, i) , where $k \in \{1, 2, 3\} \setminus \{i, j\}$. If $(i', j') = (j, i)$, then $(h_i^{-1}h_j)^2 = 1$ and since the group is torsion-free, $h_i = h_j$, a contradiction. If $(i', j') = (j, k)$, then $h_i^{-1}h_k = (h_i^{-1}h_j)^2$ and if $(i', j') = (k, i)$ then $h_i^{-1}h_k = h_i^{-1}h_j$. It follows that $H := \langle h_i^{-1} \text{supp}(\alpha) \rangle = \langle h_i^{-1}h_j \rangle$ is the infinite cyclic group. It follows from Lemma 2.1 that $h_i^{-1}\alpha, \beta g_1^{-1} \in \mathbb{F}_2[H]$. Now $(h_i^{-1}\alpha)(\beta g_1^{-1}) = 0$ contradicts the fact that the group algebra of the infinite cyclic group has no zero-divisor [16]. This completes the proof. \square

If $A = \{1, 2, 3\} \times \{1, 2, \dots, n\}$, for all $(i, j) \in A$ there must be an $(i', j') \in A$ such that $i \neq i', j \neq j'$ and $h_i g_j = h_{i'} g_{j'}$ because $\alpha\beta = (\alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3)(\beta_1 g_1 + \beta_2 g_2 + \dots + \beta_n g_n) = 0$.

The Kaplansky graph of α and β over \mathbb{F} can be defined and it is denoted by $K_{\mathbb{F}}(\alpha, \beta)$. The vertex set of $K_{\mathbb{F}}(\alpha, \beta)$ is $\text{supp}(\beta)$ and two vertices g_i and g_j are adjacent, denoted by $g_i \sim g_j$, whenever $h_{i'} g_i = h_{j'} g_j$ for some distinct $i', j' \in \{1, 2, 3\}$. So, any Kaplansky graph over \mathbb{F} is derived from a possible zero divisor with support of size 3 in the group algebra of a torsion-free group over the field \mathbb{F} . In the following, we give some properties of Kaplansky graphs.

Lemma 2.4. $K_{\mathbb{F}}(\alpha, \beta) \cong K_{\mathbb{F}}(x\alpha, \beta y)$ for all $x, y \in G$.

Proof. Note that the being minimum of the support size of βy in respect to the condition $(x\alpha)(\beta y) = 0$ is obvious as the support size of β in $\alpha\beta = 0$. Note that if S is equal to the set $\{h^{-1}h' \mid h, h' \in \text{supp}(x\alpha), h \neq h'\}$, the map on G defined by $g \mapsto gy$ for all $g \in G$ is a graph isomorphism on $\text{Cay}(G, S)$. This completes the proof. \square

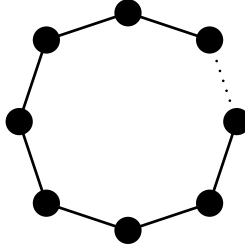


FIGURE 2. A cycle of length k in $K_{\mathbb{F}}(\alpha, \beta)$

Definition 2.5. To any cycle C of $K_{\mathbb{F}}(\alpha, \beta)$ of length k as Figure 2, we assign a $2k$ -tuple $T_C = [a_1, b_1, a_2, b_2, \dots, a_k, b_k]$, where $a_1, b_1, a_2, b_2, \dots, a_k, b_k \in \text{supp}(\alpha)$ satisfying the following relations:

$$(2.1) \quad R(T_C) : \begin{cases} a_1 g'_1 = b_1 g'_2 \\ a_2 g'_2 = b_2 g'_3 \\ \vdots \\ a_k g'_k = b_k g'_1 \end{cases}$$

where $g'_1, g'_2, g'_3, \dots, g'_k \in \text{supp}(\beta)$ are vertices of C such that $g'_i \sim g'_{i+1}$, for all $i \in \{1, 2, \dots, k-1\}$, and $g'_1 \sim g'_k$. Also, we can derive from the relations 2.1 that $r(T_C) = (a_1^{-1}b_1)(a_2^{-1}b_2) \cdots (a_k^{-1}b_k)$ is equal to

1. It follows from Lemma 2.3 that if $[a'_1, b'_1, \dots, a'_k, b'_k]$ is another $2k$ -tuple assigning to C as above, then $[a'_1, b'_1, \dots, a'_k, b'_k]$ is one of the following $2k$ -tuples:

$$\begin{aligned} &[a_1, b_1, a_2, b_2, \dots, a_{k-1}, b_{k-1}, a_k, b_k], \\ &[a_k, b_k, a_1, b_1, \dots, a_{k-2}, b_{k-2}, a_{k-1}, b_{k-1}], \\ &\quad \vdots \\ &[a_2, b_2, a_3, b_3, \dots, a_k, b_k, a_1, b_1], \\ &[b_1, a_1, b_k, a_k, \dots, b_3, a_3, b_2, a_2], \\ &[b_2, a_2, b_1, a_1, \dots, b_4, a_4, b_3, a_3], \\ &\quad \vdots \\ &[b_k, a_k, b_{k-1}, a_{k-1}, \dots, b_2, a_2, b_1, a_1]. \end{aligned}$$

The set of all such $2k$ -tuples will be denoted by $\mathcal{T}(C)$. Also, a member of the set $\mathcal{R}(C) = \{R(T_C) | T_C \in \mathcal{T}(C)\}$ is called the corresponding relations of C .

Definition 2.6. Let C be a cycle of $K_{\mathbb{F}}(\alpha, \beta)$ of length k . Since $r(T_1) = 1$ if and only if $r(T_2) = 1$, for all $T_1, T_2 \in \mathcal{T}(C)$, a member of $\{r(T_C) | T_C \in \mathcal{T}(C)\}$ is given as a representative and denoted by $r(C)$, and $r(C) = 1$ is called the relation of C .

Definition 2.7. Let C and C' be two cycles of length k in $K_{\mathbb{F}}(\alpha, \beta)$. We say that these two cycles are equivalent, if $\mathcal{T}(C) \cap \mathcal{T}(C') \neq \emptyset$.

Remark 2.8. If C and C' are two equivalent cycles of length k in $K_{\mathbb{F}}(\alpha, \beta)$, then $\mathcal{T}(C) = \mathcal{T}(C')$.

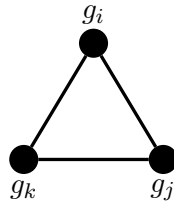


FIGURE 3. The triangle K_3 in a Kaplansky graph

Now let $\mathbb{F} = \mathbb{F}_2$. Obviously, for each (i, j) and (i, j') in A where $j \neq j'$ we have $h_i g_j \neq h_i g_{j'}$. Also, for each (i, j) and (i', j) in A where $i \neq i'$ we have $h_i g_j \neq h_{i'} g_j$. Therefore, there is a matched rectangle M corresponding to (α, β) (see [19, Definition 4.1]) that is non-degenerate and so the underlying graph $K(M)$ following [19, Definition 4.1] can be defined. We call $K(M)$, the Kaplansky graph of (α, β) over \mathbb{F}_2 and it is denoted by $K_{\mathbb{F}_2}(\alpha, \beta)$. The vertex set of the Kaplansky graph is $\text{supp}(\beta)$ and two vertices g_i and g_j are adjacent whenever $h_{i'} g_i = h_{j'} g_j$ for some distinct $i', j' \in \{1, 2, 3\}$.

The following theorem is obtained in [19].

Theorem 2.9 (Theorem 4.2 of [19]). *Any Kaplansky graph over \mathbb{F}_2 is a connected simple cubic one containing no subgraph isomorphic to a triangle.*

Proof. The proof is essentially the same as the proof of [19, Theorem 4.2], but only note that the connectedness follows from the way we have chosen β of minimum support size with respect to the property $\alpha\beta = 0$. \square

Remark 2.10. By a triangle in Theorem 2.9, we mean a subgraph such as Figure 3, where $g_i, g_j, g_k \in \text{supp}(\beta)$, with the corresponding relations as

$$(2.2) \quad \begin{cases} a_1 g_i = b_1 g_j \\ a_2 g_j = b_2 g_k \\ a_3 g_k = b_3 g_i, \\ \text{for some } a_s, b_t \in \text{supp}(\alpha) \text{ where } s, t \in \{1, 2, 3\} \\ \text{and } a_1 \neq b_1 \neq a_2 \neq b_2 \neq a_3 \neq b_3. \end{cases}$$

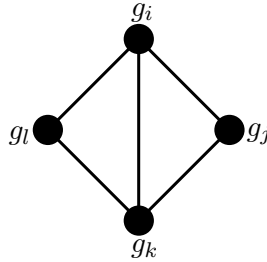


FIGURE 4. Two triangles with a common edge in the Kaplansky graph over \mathbb{F}

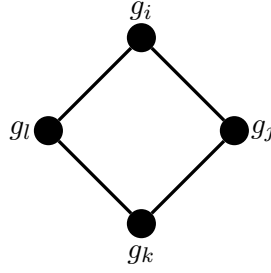
Theorem 2.11. *Kaplansky graphs over \mathbb{F} contain no subgraphs isomorphic to the graph in Figure 4 i.e. two triangles with one edge in common.*

Proof. Similar to Theorem 2.9, it can be seen that a Kaplansky graph over \mathbb{F} is also a connected simple graph containing no subgraph isomorphic to a triangle as Figure 3 with the corresponding relations 2.2. So, if $K_{\mathbb{F}}(\alpha, \beta)$ contains a subgraph isomorphic to a triangle with vertices g_i, g_j and g_k , the corresponding relations of such triangle are as follows

$$(2.3) \quad ag_i = bg_j = cg_k, \text{ where } \{a, b, c\} = \text{supp}(\alpha)$$

Suppose that $K_{\mathbb{F}}(\alpha, \beta)$ contains two triangle with one edge in common as Figure 4, where $g_i, g_j, g_k, g_l \in \text{supp}(\beta)$. With the discussion above and by the relation 2.3, $a_1 g_i = b_1 g_k = c_1 g_j$ and $a_1 g_i = b_1 g_k = c_2 g_l$ where $\{a_1, b_1, c_1\} = \{a_1, b_1, c_2\} = \text{supp}(\alpha)$. So, $c_2 g_l = a_1 g_i = b_1 g_k = c_1 g_j$ where $\{a_1, b_1, c_1\} = \{a_1, b_1, c_2\} = \text{supp}(\alpha)$. Therefore, $c_1 = c_2$ and so $g_j = g_l$, a contradiction. So, the graph $K_{\mathbb{F}}(\alpha, \beta)$ contains no two triangles with one edge in common. \square

Remark 2.12. It follows from Theorem 2.11 that if $K_{\mathbb{F}}(\alpha, \beta)$ contains an square as Figure 5, where $g_i, g_j, g_k, g_l \in \text{supp}(\beta)$, then $g_i \not\sim g_k$ and $g_j \not\sim g_l$. So, the corresponding relations of such square are as

FIGURE 5. An square in the Kaplansky graph over \mathbb{F}

follows:

$$(2.4) \quad \begin{cases} a_1 g_i = b_1 g_j \\ a_2 g_j = b_2 g_k \\ a_3 g_k = b_3 g_l \\ a_4 g_l = b_4 g_i, \\ \text{where } a_s, b_s \in \text{supp}(\alpha) \text{ for all } s \in \{1, 2, 3, 4\} \\ \text{and } a_1 \neq b_1 \neq a_2 \neq b_2 \neq a_3 \neq b_3 \neq a_4 \neq b_4 \neq a_1. \end{cases}$$

To finish this section, we consider Kaplansky graphs over \mathbb{F} containing a subgraph isomorphic to an square. We have not been able as [19, Theorem 4.2] to prove that squares are forbidden subgraphs for Kaplansky graphs even for the case that $\mathbb{F} = \mathbb{F}_2$, as triangles are so. However we show that by existence of squares, Kaplansky graphs over \mathbb{F} gives us certain slightly significant relations on elements of the support of a possible zero-divisor (see below, Theorem 2.15).

Remark 2.13. The Baumslag-Solitar group $BS(m, n)$ is the group given by the presentation $\langle a, b \mid ba^m b^{-1} = a^n \rangle$. Such groups are solvable if and only if $|m| = 1$ or $|n| = 1$ [1]. So, such latter groups and their quotients satisfy Conjecture 1.1 [16].

Theorem 2.14. *If a Kaplansky graph over an arbitrary field has an square C , then there are 9 non-equivalent cases for C which $r(C)$ is one of the relations 5, 7, 14, 17, 21, 22, 25, 26 and 29 in Table 2.*

Proof. Let C be an square as Figure 5 with the 8-tuple $T_C = [a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$ and $R(T_C)$ as 2.4. Using GAP [9], it can be seen that there are 258 cases for T_C . By Definitions 2.5 and 2.7, and by using GAP [9], it can be seen that these 258 cases can be categorized into 36 non-equivalent cases. The relations of such non-equivalent cases are listed in the column labelled by R of Table 2. It can be shown that all the relations in this table, except the relations marked by “*”s in the column labelled by E , lead to contradictions because the group G generated by h_2 and h_3 with one of such relations has at least one of the following properties:

- (1) It is an abelian group,
- (2) It is a quotient of $BS(1, k)$ or $BS(k, 1)$ where $k \in \{-2, -1, 1, 2\}$,
- (3) It has a non-trivial torsion element.

TABLE 2. The possible relations of squares in $K_{\mathbb{F}}(\alpha, \beta)$

n	R	E	n	R	E
1	$h_2^4 = 1$	T	19	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$BS(2, 1)$
2	$h_2^3 h_3 = 1$	A	20	$h_2 h_3^{-2} h_2^{-1} h_3 = 1$	$BS(1, 2)$
3	$h_2^3 h_3^{-1} h_2 = 1$	A	21	$h_2 h_3^{-3} h_2 = 1$	$*$
4	$h_2^2 h_3^2 = 1$	$BS(1, -1)$	22	$h_2 h_3^{-2} h_2 h_3 = 1$	$*$
5	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$*$	23	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$	$BS(-2, 1)$
6	$h_2^2 h_3^{-1} h_2^{-1} h_3 = 1$	$BS(1, 2)$	24	$(h_2 h_3^{-1} h_2)^2 = 1$	A
7	$h_2^2 h_3^{-2} h_2 = 1$	$*$	25	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$*$
8	$h_2^2 h_3^{-1} h_2 h_3 = 1$	$BS(1, -2)$	26	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$*$
9	$h_2 (h_2 h_3^{-1})^2 h_2 = 1$	$BS(1, -1)$	27	$(h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$BS(2, 1)$
10	$(h_2 h_3)^2 = 1$	A	28	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$BS(1, -2)$
11	$h_2 h_3 h_2 h_3^{-1} h_2 = 1$	$BS(-2, 1)$	29	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$*$
12	$h_2 h_3^3 = 1$	A	30	$(h_2 h_3^{-1})^3 h_2 = 1$	A
13	$h_2 h_3^2 h_2^{-1} h_3 = 1$	$BS(1, -2)$	31	$h_3^4 = 1$	T
14	$h_2 h_3 h_2^{-2} h_3 = 1$	$*$	32	$h_3^3 h_2^{-1} h_3 = 1$	A
15	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$BS(2, 1)$	33	$h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$BS(1, -1)$
16	$h_2 h_3 h_2^{-1} h_3^2 = 1$	$BS(-2, 1)$	34	$(h_3 h_2^{-1} h_3)^2 = 1$	A
17	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$*$	35	$(h_3 h_2^{-1})^3 h_3 = 1$	A
18	$h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$BS(2, 1)$	36	$(h_2^{-1} h_3)^4 = 1$	A

Each relation which leads to being G an abelian group or having a non-trivial torsion element is marked by an A or a T in the column labelled by E , respectively. Also, if the group G is a quotient of $BS(1, k)$ or $BS(k, 1)$ where $k \in \{-2, -1, 1, 2\}$, we show this in the column E . Therefore, there are 9 non-equivalent cases for existence of an square in a Kaplansky graph over an arbitrary field. \square

Theorem 2.15. *If a Kaplansky graph over an arbitrary field has an square C , then there exist non-trivial group elements x and y such that $x^2 = y^3$ and either $\{1, x, y\}$ or $\{1, y, y^{-1}x\}$ is the support of a zero divisor in $\mathbb{F}[G]$.*

Proof. By Theorem 2.14, there are 9 non-equivalent cases for C which $r(C)$ is one of the relations 5, 7, 14, 17, 21, 22, 25, 26 and 29 in Table 2. So, we have the followings:

- (5) $h_2^2 h_3 h_2^{-1} h_3 = 1$: Let $x = h_2^{-1} h_3$ and $y = h_2^{-1}$. So, $x^2 = y^3$. Also, since $\alpha\beta = 0$, we have $h_2^{-1}(\alpha_1 \cdot 1 + \alpha_2 h_2 + \alpha_3 h_3)(\beta_1 g_1 + \beta_2 g_2 + \cdots + \beta_n g_n) = 0$. So, $\alpha_2 \cdot 1 + \alpha_3 x + \alpha_1 y$ is a zero divisor with the support $\{1, x, y\}$.
- (7) $h_2^3 h_3^{-2} h_2 = 1$: Let $x = h_3$ and $y = h_2$. So $x^2 = y^3$ and $\alpha_1 \cdot 1 + \alpha_2 y + \alpha_3 x$ is a zero divisor with the support $\{1, x, y\}$.
- (14) $h_2 h_3 h_2^{-2} h_3 = 1$: Let $x = h_2 h_3$ and $y = h_2$. So $x^2 = y^3$ and $\alpha_1 \cdot 1 + \alpha_2 y + \alpha_3 y^{-1}x = \alpha_1 \cdot 1 + \alpha_2 h_2 + \alpha_3 h_3$ is a zero divisor with the support $\{1, y, y^{-1}x\}$.
- (17) $h_2 (h_3 h_2^{-1})^2 h_3 = 1$: Let $x = h_2^{-1}$ and $y = h_3 h_2^{-1}$. So $x^2 = y^3$. Also, since $\alpha\beta = 0$, we have $\alpha h_2^{-1} h_2 \beta = 0$. So, $\alpha_2 \cdot 1 + \alpha_1 x + \alpha_3 y$ is a zero divisor with the support $\{1, x, y\}$.

- (21) $h_2h_3^{-3}h_2 = 1$: By interchanging h_2 and h_3 in (7) and with the same discussion, the statement is true.
- (22) $h_2h_3^{-2}h_2h_3 = 1$: By interchanging h_2 and h_3 in (14) and with the same discussion, the statement is true.
- (25) $h_2h_3^{-1}h_2h_3^2 = 1$: By interchanging h_2 and h_3 in (5) and with the same discussion, the statement is true.
- (26) $h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$: Let $x = h_2h_3^{-1}h_2$ and $y = h_3^{-1}h_2$. So, $x^2 = y^3$. Also, since $\alpha_1 \cdot 1 + \alpha_2h_2 + \alpha_3h_3 = 1 + xy^{-1} + xy^{-2} = \alpha_2xy^{-1} + \alpha_3xy^{-2} + \alpha_1x^2y^{-3}$, we have $x^{-1}(\alpha_2xy^{-1} + \alpha_3xy^{-2} + \alpha_1x^2y^{-3})yy^{-1}\beta = 0$. Therefore, $\alpha_2 \cdot 1 + \alpha_3y^{-1} + \alpha_1xy^{-2}$ is also a zero divisor with the support of size 3. Furthermore, $(\alpha_3 \cdot 1 + \alpha_2y + \alpha_1y^{-1}x)(y^{-3}\beta) = (y^{-1}(\alpha_2 \cdot 1 + \alpha_3y^{-1} + \alpha_1xy^{-2})y^2)(y^{-3}\beta) = 0$. Hence, $\alpha_3 \cdot 1 + \alpha_2y + \alpha_1y^{-1}x$ is a zero divisor with the support $\{1, y, y^{-1}x\}$.
- (29) $(h_2h_3^{-1})^2h_2h_3 = 1$: By interchanging h_2 and h_3 in (17) and with the same discussion, the statement is true.

□

In the following, we discuss about the existence of two squares in $K_{\mathbb{F}}(\alpha, \beta)$.

Lemma 2.16. *Suppose that there exist two squares in $K_{\mathbb{F}}(\alpha, \beta)$. Then, such cycles are equivalent.*

Proof. By Theorem 2.14, if there exist two squares in $K_{\mathbb{F}}(\alpha, \beta)$, then these two cycles must be between 9 non-equivalent cases with the relations 5, 7, 14, 17, 21, 22, 25, 26 or 29 in Table 2. We may choose two relations similar to or different from each other. When choosing two relations different from each other, there are $\binom{9}{2} = 36$ cases. Using GAP [9], each group with two generators h_2 and h_3 , and two relations which is between 36 latter cases is finite and solvable, that is a contradiction with the assumptions. So by Theorem 2.14, if there exist two squares in the graph $K_{\mathbb{F}}(\alpha, \beta)$, then such cycles must be equivalent. □

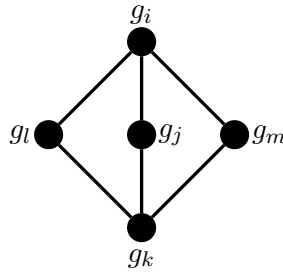


FIGURE 6. The complete bipartite graph $K_{2,3}$ in the Kaplansky graph over \mathbb{F}

Theorem 2.17. *The Kaplansky graph over \mathbb{F} contains no subgraph isomorphic to the complete bipartite graph $K_{2,3}$.*

Proof. Suppose that $K_{\mathbb{F}}(\alpha, \beta)$ contains $K_{2,3}$ as a subgraph. So, it contains 2 squares C and C' with two edges in common as Figure 6, where $g_i, g_j, g_k, g_l, g_m \in \text{supp}(\beta)$. Let $T_C = [a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$

and $T_{C'} = [a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4]$ be the 8-tuples of C and C' , respectively, with the corresponding relations as follows:

$$(2.5) \quad R(T_C) : \begin{cases} a_1 g_i = b_1 g_j \\ a_2 g_j = b_2 g_k \\ a_3 g_k = b_3 g_l \\ a_4 g_l = b_4 g_i \end{cases} \quad R(T_{C'}) : \begin{cases} a_1 g_i = b_1 g_j \\ a_2 g_j = b_2 g_k \\ a'_3 g_k = b'_3 g_m \\ a'_4 g_m = b'_4 g_i \end{cases}$$

where $a_s, b_s, a'_t, b'_t \in \text{supp}(\alpha)$ for all $s \in \{1, 2, 3, 4\}$ and $t \in \{3, 4\}$.

By Remark 2.12, we have the following inequalities.

$$(2.6) \quad a_1 \neq b_1 \neq a_2 \neq b_2 \neq a_3 \neq b_3 \neq a_4 \neq b_4 \neq a_1 \text{ and } b_2 \neq a'_3 \neq b'_3 \neq a'_4 \neq b'_4 \neq a_1.$$

Since the graph with the set of vertices $\{g_i, g_m, g_k, g_l\}$ in $K_{2,3}$ is also an square, by Remark 2.12 we have the following inequalities,

$$(2.7) \quad a_3 \neq a'_3.$$

$$(2.8) \quad b_4 \neq b'_4.$$

By Lemma 2.16, the cycles C and C' are equivalent. So, $T_{C'}$ must be in $\mathcal{T}(C)$. In the following, we show that this leads to contradictions.

- (1) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$: So $a_3 = a'_3$, that is a contradiction with the relation 2.7.
- (2) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [a_4, b_4, a_1, b_1, a_2, b_2, a_3, b_3]$: Therefore, $T_C = [a_1, b_1, a_1, b_1, a_3, b_3, a_1, b_1]$ and $T_{C'} = [a_1, b_1, a_1, b_1, a_1, b_1, a_3, b_3]$. By the relations 2.7 and 2.8 we have $a_3 \neq a_1$ and $b_3 \neq b_1$. Also, in such 8-tuples we have $a_3 \neq b_1$, $b_3 \neq a_1$ and $a_1 \neq b_1$. Therefore, $a_3 = b_3$ since $a_3, b_3 \in \text{supp}(\alpha)$, that is a contradiction.
- (3) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [a_3, b_3, a_4, b_4, a_1, b_1, a_2, b_2]$: So $a_3 = a_1 = a'_3$, that is a contradiction with the relation 2.7.
- (4) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [a_2, b_2, a_3, b_3, a_4, b_4, a_1, b_1]$: Therefore, $T_C = [a_1, b_1, a_1, b_1, a_1, b_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a_1, b_1, a_4, b_4, a_1, b_1]$. By the relations 2.7 and 2.8 we have $a_4 \neq a_1$ and $b_4 \neq b_1$. Also, in such 8-tuples we have $a_4 \neq b_1$, $b_4 \neq a_1$ and $a_1 \neq b_1$. Therefore, $a_4 = b_4$ since $a_4, b_4 \in \text{supp}(\alpha)$, that is a contradiction.
- (5) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [b_1, a_1, b_4, a_4, b_3, a_3, b_2, a_2]$: So $a_1 = b_1$, that is a contradiction with the relations 2.6.
- (6) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [b_2, a_2, b_1, a_1, b_4, a_4, b_3, a_3]$: So $b_1 = a_2$, that is a contradiction with the relations 2.6.
- (7) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [b_3, a_3, b_2, a_2, b_1, a_1, b_4, a_4]$: So $a_2 = b_2$, that is a contradiction with the relations 2.6.
- (8) $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4] = [b_4, a_4, b_3, a_3, b_2, a_2, b_1, a_1]$: So $b_2 = a_3$, that is a contradiction with the relations 2.6.

So, the graph $K_{\mathbb{F}}(\alpha, \beta)$ contains no subgraph isomorphic to the graph $K_{2,3}$. □

3. Possible zero divisors with supports of size 3 in $\mathbb{F}[G]$

If $A = \{1, 2, 3\} \times \{1, 2, \dots, n\}$, then for all $(i, j) \in A$ there must be an $(i', j') \in A$ such that $i \neq i'$, $j \neq j'$ and $h_i g_j = h_{i'} g_{j'}$ because $\alpha\beta = (\alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3)(\beta_1 g_1 + \beta_2 g_2 + \dots + \beta_n g_n) = 0$. Also, $n \geq 3$ [19]. Firstly in this section we show that $n > 3$. Then, we examine some small positive integers greater than 3 as the possible values of n and show that n must be at least 10.

3.1. The support of β cannot be of size 3. Let $|supp(\beta)| = 3$. Since $\alpha\beta = 0$, we must have $(\alpha_1\beta_1 h_1 g_1 + \alpha_1\beta_2 h_1 g_2 + \alpha_1\beta_3 h_1 g_3) + (\alpha_2\beta_1 h_2 g_1 + \alpha_2\beta_2 h_2 g_2 + \alpha_2\beta_3 h_2 g_3) + (\alpha_3\beta_1 h_3 g_1 + \alpha_3\beta_2 h_3 g_2 + \alpha_3\beta_3 h_3 g_3) = 0$. Therefore, $h_1 g_1 = h_i g_j$ for some $(i, j) \in A$ where $i \neq 1$ and $j \neq 1$. Also, $h_2 g_1 = h_{i'} g_{j'}$ for some $(i', j') \in A$ where $i' \neq 2$ and $j' \neq 1$. Furthermore, $h_3 g_1 = h_{i''} g_{j''}$ for some $(i'', j'') \in A$ where $i'' \neq 3$ and $j'' \neq 1$. Note that $j' \neq j$, $j'' \neq j$ and $j'' \neq j'$ because the Lemma 2.3 states that the set $S = \{h^{-1}h' \mid h, h' \in supp(\alpha), h \neq h'\}$ has size 6. Hence, $g_{j''} \in supp(\beta)$ and $g_{j''} \notin \{g_1, g_2, g_3\}$, a contradiction. So, $|supp(\beta)|$ must be at least 4.

3.2. The support of β must be of size greater than or equal to 10. Abelian groups satisfy Conjecture 1.1. So, G must be a nonabelian torsion-free group. The following theorem is obtained in [10].

Theorem 3.1 (Corollary 11 of [10]). *If C is a finite generating subset of a nonabelian torsion-free group G such that $1 \in C$ and $|C| \geq 4$, then $|BC| \geq |B| + |C| + 1$ for all $B \subset G$ with $|B| \geq 3$.*

Without loss of generality we may assume that G is generated by $supp(\alpha) \cup supp(\beta)$, since otherwise we replace G by the subgroup generated by this set. Also by Lemma 2.1, $\langle h_i^{-1} supp(\alpha) \rangle = \langle supp(\beta) g_j^{-1} \rangle$ for all $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \dots, n\}$. Therefore by Theorem 3.1, $|supp(\alpha)supp(\beta)| \geq |supp(\alpha)| + |supp(\beta)| + 1$. Hence, $3n \geq |supp(\alpha)supp(\beta)| \geq 4 + n$.

Theorem 3.2 (Proposition 4.12 of [5]). *There exist no $\gamma, \delta \in \mathbb{F}_2[G]$ such that $\gamma\delta = 1$, where $|supp(\gamma)| = 3$ and $|supp(\delta)| \geq 13$ is an odd integer.*

- (1) Let $n = 4$. Then with the discussion above $12 \geq |supp(\alpha)supp(\beta)| \geq 8$. Since $12 - 8 = 4$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$ for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\beta)|$ must be at least 5.
- (2) Let $n = 5$. Then with the discussion above $15 \geq |supp(\alpha)supp(\beta)| \geq 9$. Since $15 - 9 = 6$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$ for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\beta)|$ must be at least 6.
- (3) Let $n = 6$. Then with the discussion above $18 \geq |supp(\alpha)supp(\beta)| \geq 10$. Since $18 - 10 = 8$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$ for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\beta)|$ must be at least 7.
- (4) Let $n = 7$. Then with the discussion above $21 \geq |supp(\alpha)supp(\beta)| \geq 11$. Since $21 - 11 = 10$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$ for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\beta)|$ must be at least 8.
- (5) Let $n = 8$. Then with the discussion above $24 \geq |supp(\alpha)supp(\beta)| \geq 12$. Let $|supp(\alpha)supp(\beta)| > 12$. Then $|supp(\alpha)supp(\beta)| \geq 13$. Since $24 - 13 = 11$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$

for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\alpha)supp(\beta)| = 12$ and because $\alpha\beta = 0$, there is a partition π of A with all sets containing two elements, such that if (i, j) and (i', j') belong to the same set of π , then $h_i g_j = h_{i'} g_{j'}$. Let $\alpha' = \sum_{a \in supp(\alpha)} a$ and $\beta' = \sum_{b \in supp(\beta)} b$. So, $\alpha', \beta' \in \mathbb{F}_2[G]$, $|supp(\alpha')| = 3$ and $|supp(\beta')| = 8$ and with the above discussion we have $\alpha'\beta' = 0$, that is a contradiction (see below, Corollary 6.2). Therefore, $|supp(\alpha)supp(\beta)| \neq 12$ and so $|supp(\beta)|$ must be at least 9.

- (6) Let $n = 9$. Then with the discussion above $27 \geq |supp(\alpha)supp(\beta)| \geq 13$. Let $|supp(\alpha)supp(\beta)| > 13$. Then $|supp(\alpha)supp(\beta)| \geq 14$. Since $27 - 14 = 13$, there is an $(i, j) \in A$ such that $h_i g_j \neq h_{i'} g_{j'}$ for all $(i', j') \in A$ where $i \neq i'$ and $j \neq j'$, a contradiction with $\alpha\beta = 0$. So, $|supp(\alpha)supp(\beta)| = 13$ and because $\alpha\beta = 0$, there is a partition π of A with one set of size 3 and all other sets containing two elements, such that if (i, j) and (i', j') belong to the same set of π , then $h_i g_j = h_{i'} g_{j'}$. With the discussion above, $\left(\sum_{a \in supp(\alpha)} a\right) \left(\sum_{b \in supp(\beta)} b\right) x^{-1} = 1$ where $x = h_i g_j$ for some (i, j) belongs to the set of size 3 in π . Hence, there are $\gamma, \delta \in \mathbb{F}_2[G]$ such that $\gamma\delta = 1$, where $\gamma = \sum_{a \in supp(\alpha)} a$, $\delta = \sum_{b \in supp(\beta)} b x^{-1}$, $|supp(\gamma)| = 3$ and $|supp(\delta)| = |supp(\beta)| = 9$, that is a contradiction with Theorem 3.2. Therefore, $|supp(\alpha)supp(\beta)| \neq 13$ and so $|supp(\beta)|$ must be at least 10.

With the discussion above, we have the following theorem.

Theorem 3.3. *Let α and β be non-zero elements of the group algebra of any torsion-free group over an arbitrary field. If $|supp(\alpha)| = 3$ and $\alpha\beta = 0$ then $|supp(\beta)| \geq 10$.*

Proposition 3.4. *If $\mathbb{F}[G]$ has no non-zero element α with $|supp(\alpha)| \leq k$ such that $\alpha^2 = 0$, then there exist no non-zero elements $\alpha_1, \alpha_2 \in \mathbb{F}[G]$ such that $\alpha_1\alpha_2 = 0$ and $|supp(\alpha_1)||supp(\alpha_2)| \leq k$.*

Proof. Suppose, for a contradiction, that $\alpha_1, \alpha_2 \in \mathbb{F}[G] \setminus \{0\}$ such that $\alpha_1\alpha_2 = 0$ and $|supp(\alpha_1)||supp(\alpha_2)| \leq k$. We may assume that $1 \in supp(\alpha_1) \cap supp(\alpha_2)$, since $(a^{-1}\alpha_1)(\alpha_2 b^{-1}) = 0$ for any $a \in supp(\alpha_1)$ and $b \in supp(\alpha_2)$.

Suppose, for a contradiction, that $\alpha_2 x \alpha_1 = 0$ for all $x \in G$. Then it follows from [17, Lemma 1.3, p. 3] that $\theta(\alpha_2)\theta(\alpha_1) = 0$, where θ is the projection $\theta : \mathbb{F}[G] \rightarrow \mathbb{F}[\Delta]$ given by $\beta = \sum_{x \in G} f_x x \mapsto \theta(\beta) = \sum_{x \in \Delta} f_x x$, where Δ is the subgroup of all elements of G having a finite number of conjugates in G (see [17, p. 3]). Now it follows from [17, Lemma 2.2, p. 5] and [17, Lemma 2.4, p. 6] that $\theta(\alpha_1) = 0$ or $\theta(\alpha_2) = 0$, which are both contradiction since $1 \in supp(\alpha_1) \cap supp(\alpha_2)$. Therefore, there exists an element $x \in G$ such that $\beta = \alpha_2 x \alpha_1 \neq 0$. Now

$$\beta^2 = (\alpha_2 x \alpha_1)^2 = \alpha_2 x \alpha_1 \alpha_2 x \alpha_1 = 0$$

and

$$|supp(\beta)| \leq |supp(\alpha_2)||supp(x\alpha_1)| = |supp(\alpha_2)||supp(\alpha_1)| \leq k,$$

which is a contradiction. This completes the proof. \square

In the next three sections, we discuss about Kaplansky graphs over \mathbb{F}_2 and give some forbidden subgraphs for such graphs.

4. Kaplansky graphs over \mathbb{F}_2 and some of their subgraphs containing an square

Throughout the rest of this paper, except in Section 7, let \mathbb{F} be the finite field \mathbb{F}_2 and $\alpha = h_1 + h_2 + h_3 \in \mathbb{F}_2[G]$ such that $|supp(\alpha)| = 3$. Suppose further that $\alpha\beta = 0$ for some non-zero $\beta \in \mathbb{F}_2[G]$ and assume that $n := |supp(\beta)|$ is minimum with respect to the latter property. Let $\beta = g_1 + g_2 + \cdots + g_n$. If there is no ambiguity, we denote the Kaplansky graph of (α, β) over \mathbb{F}_2 by $K(\alpha, \beta)$ and simply call it Kaplansky graph.

Lemma 4.1. *The Kaplansky graph $K(\alpha, \beta)$ is isomorphic to the induced subgraph on the set $supp(\beta)$ of the Cayley graph $Cay(G, S)$, where $S = \{h^{-1}h' \mid h, h' \in supp(\alpha), h \neq h'\}$.*

Proof. Let M be the matched rectangle corresponding to (α, β) (see [19, Definition 4.1]). The vertex set of $K(M) = K(\alpha, \beta)$ is the columns of M which are labelled by the elements of $supp(\beta)$ and two distinct columns c and c' are adjacent whenever their labels g and $g' \in supp(\beta)$ respectively, satisfying $hg = h'g'$ for some $h, h' \in supp(\alpha)$; Or equivalently the columns c and c' are adjacent whenever $gg'^{-1} \in S$. Hence the map ψ from the columns of M to $supp(\beta)$ which sends each column to its label is a graph isomorphism from $K(M) = K(\alpha, \beta)$ to the induced subgraph on the set $supp(\beta)$ of the Cayley graph $Cay(G, S)$. \square

Remark 4.2. It follows from Lemma 4.1 and Theorem 2.9 that the induced subgraph of $Cay(G, S)$ on $supp(\beta)$ is a cubic graph having no subgraph isomorphic to a triangle. Also $n = |supp(\beta)|$ which is the number of vertices of $K(\alpha, \beta)$ is always an even number, since the number of vertices of any cubic graph is even.

In the rest of this section, we consider Kaplansky graphs containing a subgraph isomorphic to an square.

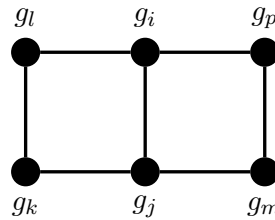


FIGURE 7. Two squares with one common edge in $K(\alpha, \beta)$

Theorem 4.3. *Suppose that $K(\alpha, \beta)$ contains two squares with exactly one edge in common. Then exactly one of the relations 14, 22 or 26 of Table 2 will be satisfied in G .*

Proof. Suppose that the graph $K(\alpha, \beta)$ contains two squares C and C' with exactly one common edge as Figure 7, where $g_i, g_j, g_k, g_l, g_m, g_p \in supp(\beta)$. Let $T_C = [a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4]$ be the 8-tuples of C and C' , respectively, with the corresponding relations

as follows:

$$(4.1) \quad R(T_C) : \begin{cases} a_1 g_i = b_1 g_j \\ a_2 g_j = b_2 g_k \\ a_3 g_k = b_3 g_l \\ a_4 g_l = b_4 g_i \end{cases} \quad R(T_{C'}) : \begin{cases} a_1 g_i = b_1 g_j \\ a'_2 g_j = b'_2 g_m \\ a'_3 g_m = b'_3 g_p \\ a'_4 g_p = b'_4 g_i \end{cases}$$

where $a_z, b_z, a'_t, b'_t \in \text{supp}(\alpha)$ for all $z \in \{1, 2, 3, 4\}$ and $t \in \{2, 3, 4\}$.

By Remark 2.12, $a_1 \neq b_1 \neq \dots \neq a_4 \neq b_4 \neq a_1$ and $b_1 \neq a'_2 \neq \dots \neq a'_4 \neq b'_4 \neq a_1$. We want to prove that $a_2 \neq a'_2$ and $b_4 \neq b'_4$. Suppose that $a_2 = a'_2$. So we have $a_2 g_j = a'_2 g_j$. Since $a_2 g_j = b_2 g_k$ and $a'_2 g_j = b'_2 g_m$, we have $b_2 g_k = a'_2 g_j = b'_2 g_m$. Since $(1 + h_2 + h_3)(g_1 + g_2 + \dots + g_n) = 0$ in $\mathbb{F}_2[G]$, there must exist $g_a \in \{g_1, g_2, \dots, g_n\} \setminus \{g_j, g_k, g_m\}$ and $h_a \in \text{supp}(\alpha)$ such that $b_2 g_k = a'_2 g_j = b'_2 g_m = h_a g_a$. Since the set $\text{supp}(\alpha)$ has size 3 and $\{a'_2, b_2, b'_2\} = \text{supp}(\alpha)$, we have $h_a \in \{a'_2, a_3, a'_3\}$. Without loss of generality we may assume that $h_a = a'_2$. So $g_j = g_a$, that is a contradiction. Hence,

$$(4.2) \quad a_2 \neq a'_2.$$

Also with the same discussion such as above, we have

$$(4.3) \quad b_4 \neq b'_4.$$

By Lemma 2.16, the cycles C and C' are equivalent. So, $T_{C'}$ must be in $\mathcal{T}(C)$. In the following, we discuss about all the possible cases in details.

- (1) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4]$: So $a_2 = a'_2$, that is a contradiction with the relation 4.2.
- (2) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [a_4, b_4, a_1, b_1, a_2, b_2, a_3, b_3]$: So, $T_C = [a_1, b_1, a_2, b_2, a_3, b_3, a_1, b_1]$ and $T_{C'} = [a_1, b_1, a_1, b_1, a_2, b_2, a_3, b_3]$. By the relations 4.1, 4.2 and 4.3, we have $a_2 \neq a_1$, $a_2 \neq b_1$, $b_3 \neq a_1$, $b_3 \neq b_1$ and $a_1 \neq b_1$. Therefore, $a_2 = b_3$ because $a_2, b_3 \in \text{supp}(\alpha)$. So, $T_C = [a_1, b_1, a_2, b_2, a_3, a_2, a_1, b_1]$ and $T_{C'} = [a_1, b_1, a_1, b_1, a_2, b_2, a_3, a_2]$. Since by the relation 4.1, $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$, $b_2 \neq a_2$, $a_3 \neq a_2$ and $b_2 \neq a_3$, there are just two cases for choosing b_2 and a_3 . In the following, we show that both of them lead to contradictions.
 - i) $b_2 = a_1$ and $a_3 = b_1$: So, $T_C = [a_1, b_1, a_2, a_1, b_1, a_2, a_1, b_1]$ and the relation of such cycle is $a_1^{-1} b_1 a_2^{-1} a_1 b_1^{-1} a_2 a_1^{-1} b_1 = 1$. Since $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$, just one of the following cases may be happened:
 - a) $a_1 = 1$: $a_2^{-1} b_1 a_2 = b_1^2$ and so $\langle h_2, h_3 \rangle = BS(1, 2)$, that is a contradiction.
 - b) $b_1 = 1$: $a_2^{-1} a_1 a_2 = a_1^2$ and so $\langle h_2, h_3 \rangle = BS(1, 2)$, that is a contradiction.
 - c) $a_2 = 1$: $a_1^{-1} b_1 a_1 b_1^{-1} a_1^{-1} b_1 = 1$. If $x = a_1^{-1} b_1$ and $y = b_1^{-1}$, then $y^{-1} x y = x^2$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(1, 2)$, that is a contradiction.
 - ii) $b_2 = b_1$ and $a_3 = a_1$: So, $T_C = [a_1, b_1, a_2, b_1, a_1, a_2, a_1, b_1]$ and the relation of such cycle is $a_1^{-1} b_1 a_2^{-1} b_1 a_1^{-1} a_2 a_1^{-1} b_1 = 1$. With the same discussion as in item (i), exactly one of the following cases may be happened:
 - a) $a_1 = 1$: $a_2^{-1} b_1 a_2 = b_1^{-2}$ and so $\langle h_2, h_3 \rangle = BS(1, -2)$, that is a contradiction.
 - b) $b_1 = 1$: $a_2^{-1} a_1 a_2 = a_1^{-2}$ and so $\langle h_2, h_3 \rangle = BS(1, -2)$, that is a contradiction.

- c) $a_2 = 1$: $a_1^{-1}b_1^2a_1^{-2}b_1 = 1$. If $x = b_1^{-1}a_1$ and $y = b_1^{-1}$, then $y^{-1}xy = x^{-2}$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(1, -2)$, that is a contradiction.

Hence, $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] \neq [a_4, b_4, a_1, b_1, a_2, b_2, a_3, b_3]$.

- (3) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [a_3, b_3, a_4, b_4, a_1, b_1, a_2, b_2]$: So, $T_C = [a_1, b_1, a_2, b_2, a_1, b_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a_4, b_4, a_1, b_1, a_2, b_2]$. By the relation 4.1, we have $a_2 \neq b_1$. In the following, we show that $a_2 \neq a_1$.

Suppose that $a_2 = a_1$. So, $T_C = [a_1, b_1, a_1, b_2, a_1, b_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a_4, b_4, a_1, b_1, a_1, b_2]$. By the relations 4.1, 4.2 and 4.3, we have $b_1 \neq a_1$, $b_1 \neq a_4$, $b_4 \neq a_1$, $b_4 \neq a_4$ and $a_1 \neq a_4$. Therefore, $b_1 = b_4$ since $b_1, b_4 \in \text{supp}(\alpha)$. Now by the relations 4.1, 4.2 and 4.3, we have $b_2 \neq a_1$, $b_2 \neq b_1$, $a_4 \neq a_1$, $a_4 \neq b_1$ and $a_1 \neq b_1$. Hence, $b_2 = a_4$ because $b_2, a_4 \in \text{supp}(\alpha)$. So $T_C = [a_1, b_1, a_1, b_2, a_1, b_1, b_2, b_1]$ and the relation of C is $a_1^{-1}b_1a_1^{-1}b_2a_1^{-1}b_1b_2^{-1}b_1 = 1$. Since $\{a_1, b_1, b_2\} = \text{supp}(\alpha)$, just one of the following cases may be happened:

- a) $a_1 = 1$: $b_2^{-1}b_1^{-2}b_2 = b_1$ and so $\langle h_2, h_3 \rangle = BS(-2, 1)$, that is a contradiction.
- b) $b_1 = 1$: $b_2^{-1}a_1^{-2}b_2 = a_1$ and so $\langle h_2, h_3 \rangle = BS(-2, 1)$, that is a contradiction.
- c) $b_2 = 1$: $a_1^{-1}b_1a_1^{-2}b_1^2 = 1$. If $x = b_1a_1^{-1}$ and $y = a_1^{-1}$, then $y^{-1}x^{-2}y = x$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(-2, 1)$, that is a contradiction.

Hence, $a_2 \neq a_1$. Also, by the relations 4.1, 4.2 and 4.3 and by T_C and $T_{C'}$, we have $a_1 \neq b_1$, $a_4 \neq a_2$, $a_4 \neq b_1$ and $b_1 \neq a_2$. Therefore, $a_1 = a_4$ because $a_1, a_4 \in \text{supp}(\alpha)$. Now by the relations 4.1, 4.2 and 4.3, we have $a_2 \neq a_1$, $a_2 \neq b_2$, $b_4 \neq a_1$, $b_4 \neq b_2$ and $a_1 \neq b_2$. So, $a_2 = b_4$ because $a_2, b_4 \in \text{supp}(\alpha)$. Also, $b_2 \neq a_1$, $b_2 \neq a_2$, $b_1 \neq a_1$, $b_1 \neq a_2$ and $a_1 \neq a_2$. Therefore, $b_1 = b_2$ because $b_1, b_2 \in \text{supp}(\alpha)$. Hence, $T_C = [a_1, b_1, a_2, b_1, a_1, b_1, a_1, a_2]$ and the relation of such cycle is $a_1^{-1}b_1a_2^{-1}b_1a_1^{-1}b_1a_1^{-1}a_2 = 1$. Since $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$, exactly one of the following cases may be happened:

- a) $a_1 = 1$: $a_2^{-1}b_1^{-2}a_2 = b_1$ and so $\langle h_2, h_3 \rangle = BS(-2, 1)$, that is a contradiction.
- b) $b_1 = 1$: $a_2^{-1}a_1^{-2}a_2 = a_1$ and so $\langle h_2, h_3 \rangle = BS(-2, 1)$, that is a contradiction.
- c) $a_2 = 1$: $a_1^{-1}b_1^2a_1^{-1}b_1a_1^{-1} = 1$. If $x = a_1b_1^{-1}$ and $y = a_1^{-1}$, then $y^{-1}x^{-2}y = x$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(-2, 1)$, that is a contradiction.

Hence, $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] \neq [a_3, b_3, a_4, b_4, a_1, b_1, a_2, b_2]$.

- (4) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [a_2, b_2, a_3, b_3, a_4, b_4, a_1, b_1]$: So, $T_C = [a_1, b_1, a_1, b_1, a_3, b_3, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a_3, b_3, a_4, b_4, a_1, b_1]$. By the relations 4.1, 4.2 and 4.3, we have $a_3 \neq a_1$, $a_3 \neq b_1$, $b_4 \neq a_1$, $b_4 \neq b_1$ and $a_1 \neq b_1$. Therefore, $a_3 = b_4$ because $a_3, b_4 \in \text{supp}(\alpha)$. Now because $\{a_1, b_1, a_3\} = \text{supp}(\alpha)$, $b_3 \neq a_3$, $a_4 \neq a_3$ and $b_3 \neq a_4$, there are two cases for choosing b_3 and a_4 . In the following, we show that both of them lead to contradictions.

- i) $b_3 = a_1$ and $a_4 = b_1$: So, $T_C = [a_1, b_1, a_1, b_1, a_3, a_1, b_1, a_3]$ and the relation of such cycle is $a_1^{-1}b_1a_1^{-1}b_1a_1^{-1}a_3b_1^{-1}a_3 = 1$. Since $\{a_1, b_1, a_3\} = \text{supp}(\alpha)$, exactly one of the following cases may be happened:

- a) $a_1 = 1$: $a_3^{-1}b_1a_3 = b_1^2$ and so $\langle h_2, h_3 \rangle = BS(1, 2)$, that is a contradiction.
- b) $b_1 = 1$: $a_3^{-1}a_1a_3 = a_1^2$ and so $\langle h_2, h_3 \rangle = BS(1, 2)$, that is a contradiction.
- c) $a_3 = 1$: $a_1^{-1}b_1a_1b_1^{-1}a_1^{-1}b_1 = 1$. If $x = a_1^{-1}b_1$ and $y = b_1^{-1}$, then $y^{-1}xy = x^2$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(1, 2)$, that is a contradiction.

ii) $b_3 = b_1$ and $a_4 = a_1$: So, $T_C = [a_1, b_1, a_1, b_1, a_3, b_1, a_1, a_3]$ and relation of such cycle is $a_1^{-1}b_1a_1^{-1}b_1a_3^{-1}b_1a_1^{-1}a_3 = 1$. With the same discussion as in item (i), exactly one of the following cases may be happened:

- a) $a_1 = 1$: $a_3^{-1}b_1a_3 = b_1^{-2}$ and so $\langle h_2, h_3 \rangle = BS(1, -2)$, that is a contradiction.
- b) $b_1 = 1$: $a_3^{-1}a_1a_3 = a_1^{-2}$ and so $\langle h_2, h_3 \rangle = BS(1, -2)$, that is a contradiction.
- c) $a_3 = 1$: $a_1^{-1}b_1^2a_1^{-2}b_1 = 1$. If $x = b_1^{-1}a_1$ and $y = b_1^{-1}$, then $y^{-1}xy = x^{-2}$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(1, -2)$, that is a contradiction.

Hence, $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] \neq [a_2, b_2, a_3, b_3, a_4, b_4, a_1, b_1]$.

- (5) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [b_1, a_1, b_4, a_4, b_3, a_3, b_2, a_2]$: So $a_1 = b_1$, that is a contradiction with the relation 4.1.
- (6) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [b_4, a_4, b_3, a_3, b_2, a_2, b_1, a_1]$: So $a_1 = b_4$, that is a contradiction with the relation 4.1.
- (7) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [b_3, a_3, b_2, a_2, b_1, a_1, b_4, a_4]$: So, $T_C = [a_1, b_1, a_2, b_2, b_1, a_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, b_2, a_2, b_1, a_1, b_4, a_4]$. Since by the relation 4.1, $a_1 \neq b_1$ and $a_2 \neq b_1$, there are two cases, namely $a_2 = a_1$ and $a_2 \neq a_1$.

A) $a_2 = a_1$: So, $T_C = [a_1, b_1, a_1, b_2, b_1, a_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, b_2, a_1, b_1, a_1, b_4, a_4]$. By the relation 4.1, we have $b_2 \neq a_1$, $b_2 \neq b_1$ and $a_1 \neq b_1$. So, $\{a_1, b_1, b_2\} = \text{supp}(\alpha)$. Also by this relation, $a_4 \neq a_1$, $b_4 \neq a_1$ and $a_4 \neq b_4$. Therefore, there are two cases for choosing a_4 and b_4 which we show that one of them leads to a contradiction with our assumptions.

Suppose that $a_4 = b_2$ and $b_4 = b_1$. So, $T_C = [a_1, b_1, a_1, b_2, b_1, a_1, b_2, b_1]$ and the relation of such cycle is $a_1^{-1}b_1a_1^{-1}b_2b_1^{-1}a_1b_2^{-1}b_1 = 1$. Since $\{a_1, b_1, b_2\} = \text{supp}(\alpha)$, just one of the following cases may be happened:

- a) $a_1 = 1$: $b_2^{-1}b_1^2b_2 = b_1$ and so $\langle h_2, h_3 \rangle = BS(2, 1)$, that is a contradiction.
- b) $b_1 = 1$: $b_2^{-1}a_1^2b_2 = a_1$ and so $\langle h_2, h_3 \rangle = BS(2, 1)$, that is a contradiction.
- c) $b_2 = 1$: $a_1^{-1}b_1a_1^{-1}b_1^{-1}a_1b_1 = 1$. If $x = b_1a_1^{-1}$ and $y = a_1^{-1}$, then $y^{-1}x^2y = x$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(2, 1)$, that is a contradiction.

Therefore, if $a_2 = a_1$, then $a_4 = b_1$ and $b_4 = b_2$. So, $T_C = [a_1, b_1, a_1, b_2, b_1, a_1, b_1, b_2]$ and the relation of such cycle is $a_1^{-1}b_1a_1^{-1}b_2b_1^{-1}a_1b_1^{-1}b_2 = 1$. Since $\{a_1, b_1, b_2\} = \text{supp}(\alpha)$, exactly one of the following cases may be happened:

- a) $a_1 = 1$: $b_1b_2b_1^{-2}b_2 = 1$, where $\{b_1, b_2\} = \{h_2, h_3\}$.
- b) $b_1 = 1$: $a_1b_2a_1^{-2}b_2 = 1$, where $\{a_1, b_2\} = \{h_2, h_3\}$.
- c) $b_2 = 1$: $b_1a_1^{-1}b_1a_1b_1^{-1}a_1 = 1$, where $\{a_1, b_1\} = \{h_2, h_3\}$.

Hence, if $a_2 = a_1$, then $T_C = [a_1, b_1, a_1, b_2, b_1, a_1, b_1, b_2]$ and $T_{C'} = [a_1, b_1, b_2, a_1, b_1, a_1, b_2, b_1]$, where $\{a_1, b_1, b_2\} = \text{supp}(\alpha)$. Also with this assumption, exactly one of the relations 14, 22 or 26 of Table 2 will be satisfied in G .

B) $a_2 \neq a_1$: By the relation 4.1, we have $a_2 \neq a_1$, $a_2 \neq b_1$ and $a_1 \neq b_1$. So, $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$. Also $b_2 = a_1$ because $b_2 \neq b_1$ and $b_2 \neq a_2$, by the relation 4.1. So, $T_C = [a_1, b_1, a_2, a_1, b_1, a_1, a_4, b_4]$ and $T_{C'} = [a_1, b_1, a_1, a_2, b_1, a_1, b_4, a_4]$. Also by the relation 4.1, $a_4 \neq a_1$, $b_4 \neq a_1$ and $a_4 \neq b_4$. Therefore, there are two cases for choosing a_4 and b_4 which we show that one of them leads to a contradiction with our assumptions.

Suppose that $a_4 = b_1$ and $b_4 = a_2$. So, $T_C = [a_1, b_1, a_2, a_1, b_1, a_1, b_1, a_2]$ and the relation of such cycle is $a_1^{-1}b_1a_2^{-1}a_1b_1^{-1}a_1b_1^{-1}a_2 = 1$. Since $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$, exactly one of the following cases may be happened:

- a) $a_1 = 1$: $a_2^{-1}b_1^2a_2 = b_1$ and so $\langle h_2, h_3 \rangle = BS(2, 1)$, that is a contradiction.
- b) $b_1 = 1$: $a_2^{-1}a_1^2a_2 = a_1$ and so $\langle h_2, h_3 \rangle = BS(2, 1)$, that is a contradiction.
- c) $a_2 = 1$: $a_1^{-1}b_1a_1b_1^{-1}a_1b_1^{-1} = 1$. If $x = a_1b_1^{-1}$ and $y = b_1^{-1}$, then $y^{-1}x^2y = x$ and $\langle h_2, h_3 \rangle = \langle x, y \rangle = BS(2, 1)$, that is a contradiction.

Therefore, if $a_2 \neq a_1$, then $a_4 = a_2$ and $b_4 = b_1$. So, $T_C = [a_1, b_1, a_2, a_1, b_1, a_1, a_2, b_1]$ and the relation of such cycle is $a_1^{-1}b_1a_2^{-1}a_1b_1^{-1}a_1a_2^{-1}b_1 = 1$. Since $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$, exactly one of the following cases may be happened:

- a) $a_1 = 1$: $b_1a_2b_1^{-2}a_2 = 1$, where $\{b_1, a_2\} = \{h_2, h_3\}$.
- b) $b_1 = 1$: $a_1a_2a_1^{-2}a_2 = 1$, where $\{a_1, a_2\} = \{h_2, h_3\}$.
- c) $a_2 = 1$: $b_1a_1^{-1}b_1a_1b_1^{-1}a_1 = 1$, where $\{a_1, b_1\} = \{h_2, h_3\}$.

Hence, if $a_2 \neq a_1$, then $T_C = [a_1, b_1, a_2, a_1, b_1, a_1, a_2, b_1]$ and $T_{C'} = [a_1, b_1, a_1, a_2, b_1, a_1, b_1, a_2]$, where $\{a_1, b_1, a_2\} = \text{supp}(\alpha)$. Also with this assumption, exactly one of the relations 14, 22 or 26 of Table 2 will be satisfied in G .

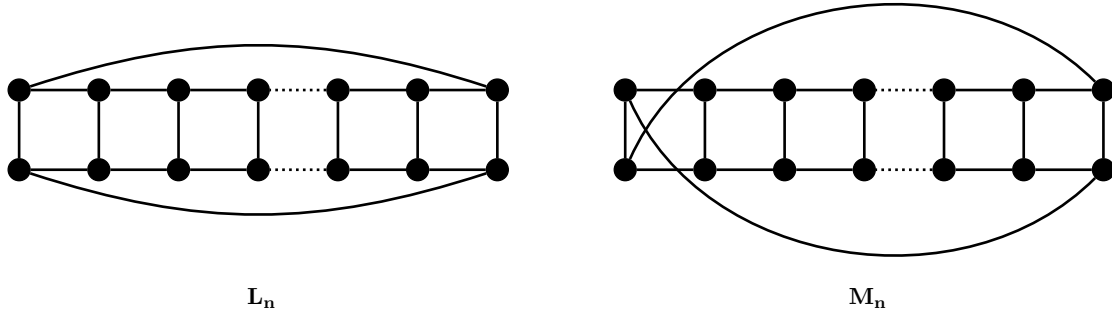
- (8) $[a_1, b_1, a'_2, b'_2, a'_3, b'_3, a'_4, b'_4] = [b_2, a_2, b_1, a_1, b_4, a_4, b_3, a_3]$: So $b_1 = a_2$, that is a contradiction with the relation 4.1.

Hence by the discussion above, if $K(\alpha, \beta)$ contains two squares with exactly one edge in common, then exactly one of the relations 14, 22 or 26 of Table 2 will be satisfied in G . \square

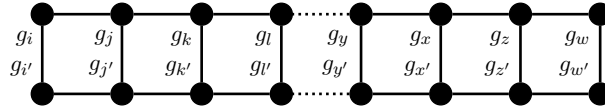
Remark 4.4. By the proof of Theorem 4.3, if the Kaplansky graph contains two squares C and C' with exactly one common edge, then the relations and the 8-tuples of such cycles must be of just one of the following forms, where the first two components of these 8-tuples are corresponded to the common edge of C and C' :

- 1) $T_C = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ and $T_{C'} = [h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$, and vice versa, with relation 14 of Table 2.
- 2) $T_C = [1, h_2, 1, h_3, h_2, 1, h_2, h_3]$ and $T_{C'} = [1, h_2, h_3, 1, h_2, 1, h_3, h_2]$, and vice versa, with relation 14 of Table 2.
- 3) $T_C = [h_3, 1, h_3, h_2, 1, h_3, 1, h_2]$ and $T_{C'} = [h_3, 1, h_2, h_3, 1, h_3, h_2, 1]$, and vice versa, with relation 22 of Table 2.
- 4) $T_C = [1, h_3, 1, h_2, h_3, 1, h_3, h_2]$ and $T_{C'} = [1, h_3, h_2, 1, h_3, 1, h_2, h_3]$, and vice versa, with relation 22 of Table 2.
- 5) $T_C = [h_2, h_3, h_2, 1, h_3, h_2, h_3, 1]$ and $T_{C'} = [h_2, h_3, 1, h_2, h_3, h_2, 1, h_3]$, and vice versa, with relation 26 of Table 2.
- 6) $T_C = [h_3, h_2, h_3, 1, h_2, h_3, h_2, 1]$ and $T_{C'} = [h_3, h_2, 1, h_3, h_2, h_3, 1, h_2]$, and vice versa, with relation 26 of Table 2.

Theorem 4.5. *The Kaplansky graph $K(\alpha, \beta)$ is isomorphic to none of the graphs L_n and M_n from Figure 8.*

FIGURE 8. Two graphs which are not isomorphic to $K(\alpha, \beta)$

Proof. If n is the number of vertices of the graph L_n or M_n , then it can be seen in Figure 8 that the number of cycles of length 4 in L_n or M_n is equal to $\frac{n}{2}$, and each two consecutive C_4 cycles have a common edge. So, if $K(\alpha, \beta)$ contains no two C_4 cycles with exactly one common edge, then it cannot be isomorphic to the graphs L_n and M_n . Suppose that the graph $K(\alpha, \beta)$ contains two cycles of length 4 with exactly one edge in common.

FIGURE 9. Consecutive cycles of length 4 in the graph $K(\alpha, \beta)$

Suppose that $K(\alpha, \beta)$ contains a subgraph as like as Figure 9, whose number of consecutive C_4 cycles is denoted by m . We denote these cycles by C_1, C_2, \dots, C_{m-1} and C_m from the left to the right. For all $i \in \{1, 2, \dots, m-1\}$, the 8-tuples of C_i and C_{i+1} must be of exactly one of the 6 cases in Remark 4.4, where the first two components are corresponded to their common edge. In the following, we prove the statement of the theorem for the case that $C_1 = C$ and $C_2 = C'$ in the first item of Remark 4.4. The other 5 cases can be proven similarly.

Suppose that $C_1 = C$, $C_2 = C'$ are the same as the first item in Remark 4.4, i.e. $T_{C_1} = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ and $T_{C_2} = [h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$ such that

$$(4.4) \quad R(T_{C_1}) : \begin{cases} h_2 g_j = g_{j'} \\ h_2 g_{j'} = h_3 g_{i'} \\ g_{i'} = h_2 g_i \\ g_i = h_3 g_j \end{cases} \quad R(T_{C_2}) : \begin{cases} h_2 g_j = g_{j'} \\ h_3 g_{j'} = h_2 g_{k'} \\ g_{k'} = h_2 g_k \\ h_3 g_k = g_j \end{cases}$$

where $\{g_j, g_{j'}, g_{i'}, g_i\}$ and $\{g_j, g_{j'}, g_{k'}, g_k\}$ are the vertex sets of C_1 and C_2 , respectively.

Using induction on m , we show that for all $i \in \{1, 2, \dots, m-1\}$, the corresponding relations of C_i and C_{i+1} are the same as the relations 4.4, where the first relation of each part is related to the common edge between C_i and C_{i+1} and the relations of these cycles are written clockwise and counter clockwise, respectively.

If $m = 2$, then the above statement is obviously true. Suppose that this statement is true for $m - 1$. So, $T_{C_{m-2}} = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ and $T_{C_{m-1}} = [h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$, where the first two components are related to the common edge between these cycles. Suppose that $\{g_x, g_{x'}, g_{y'}, g_y\}$ and $\{g_x, g_{x'}, g_{z'}, g_z\}$ are the vertex sets of C_{m-2} and C_{m-1} , respectively, and the corresponding relations of these cycles are as relations 4.4, by replacing $g_j, g_{j'}, g_{i'}, g_i, g_{k'}$ and g_k with $g_x, g_{x'}, g_{y'}, g_y, g_{z'}$ and g_z , respectively. Obviously, we may rewrite the relations of C_{m-1} such that the first relation being $g_{z'} = h_2 g_z$. So, there is $T'_{C_{m-1}} \in \mathcal{T}(C_{m-1})$ equal to $T_{C_1} = T_C = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ with the corresponding relations as follows:

$$R(T'_{C_{m-1}}) : \begin{cases} h_2 g_z = g_{z'} \\ h_2 g_{z'} = h_3 g_{x'} \\ g_{x'} = h_2 g_x \\ g_x = h_3 g_z \end{cases}$$

Therefore by Remark 4.4, there is $T_{C_m} \in \mathcal{T}(C_m)$ equal to $[h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$ with the corresponding relations

$$R(T_{C_m}) : \begin{cases} h_2 g_z = g_{z'} \\ h_3 g_{z'} = h_2 g_{w'} \\ g_{w'} = h_2 g_w \\ h_3 g_w = g_z \end{cases}$$

where $\{g_z, g_{z'}, g_{w'}, g_w\}$ is the vertex set of C_m .

Hence, for all $i \in \{1, 2, \dots, m - 1\}$, the corresponding relations of C_i and C_{i+1} are the same as the relations 4.4, where the first relation of each part is related to the common edge between C_i and C_{i+1} and the relations of these cycles are written clockwise and counter clockwise, respectively.

Now suppose that $K(\alpha, \beta)$ is isomorphic to the graph L_n , where the vertex set of $K(\alpha, \beta)$ is equal to the set $B = \{g_1, g_2, \dots, g_n\}$. Also suppose that the vertices of L_n in the top row and the bottom row are $g'_1, g'_2, g'_3, g'_4, \dots, g'_{\frac{n}{2}-2}, g'_{\frac{n}{2}-1}, g'_{\frac{n}{2}}$ and $g'_n, g'_{n-1}, g'_{n-2}, g'_{n-3}, \dots, g'_{\frac{n}{2}+3}, g'_{\frac{n}{2}+2}, g'_{\frac{n}{2}+1}$, respectively from the left to the right, where $\{g'_a | a \in \{1, 2, \dots, n\}\} = \{g_1, g_2, \dots, g_n\}$. Let $m = \frac{n}{2}$ be the number of cycles of length 4 in L_n that each two of them which are consecutive have a common edge. We denote these cycles with C_1, C_2, \dots, C_{m-1} and C_m , where C_1 and C_2 are the cycles with vertex sets $\{g'_1, g'_2, g'_{n-1}, g'_n\}$ and $\{g'_2, g'_3, g'_{n-2}, g'_{n-1}\}$, respectively. In addition, Suppose that $C_1 = C$, $C_2 = C'$ are the same as the first item in Remark 4.4, i.e. the 8-tuples of C_1 and C_2 are $[h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ and $[h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$, respectively, and the corresponding relations of these cycles are as relations 4.4, by replacing $g_j, g_{j'}, g_{i'}, g_i, g_{k'}$ and g_k with $g'_2, g'_{n-1}, g'_n, g'_1, g'_{n-2}$ and g'_3 , respectively.

With the discussion above, for all $i \in \{1, 2, \dots, m - 1\}$, the corresponding relations of C_i and C_{i+1} are the same as the relations of $C_1 = C$, $C_2 = C'$, when the first relation of each part is related to the common edge between C_i and C_{i+1} and the relations of these cycles are written clockwise and counter clockwise, respectively. Therefore we have $g'_1 = h_3 g'_2, g'_2 = h_3 g'_3, \dots, g'_{\frac{n}{2}-2} = h_3 g'_{\frac{n}{2}-1}, g'_{\frac{n}{2}-1} = h_3 g'_{\frac{n}{2}}$ and $g'_{\frac{n}{2}} = h_3 g'_1$. So, $g'_1 = h_3^{\frac{n}{2}} g'_1$ and hence $h_3^{\frac{n}{2}} = 1$. Since G is a torsion-free group, we have $h_3 = 1$ that is a contradiction with $|supp(\alpha)| = 3$. Therefore, with above assumptions on C_1 and C_2 , the graph $K(\alpha, \beta)$ cannot be isomorphic to the graph L_n .

Now suppose that $K(\alpha, \beta)$ is isomorphic to the graph M_n , where the vertex set of $K(\alpha, \beta)$ is equal to the set $B = \{g_1, g_2, \dots, g_n\}$. With the same assumptions such as above on the cycles of length 4 in M_n and by using the corresponding relations of these cycles, we have $g'_1 = h_3 g'_2, g'_2 = h_3 g'_3, \dots, g'_{\frac{n}{2}-2} = h_3 g'_{\frac{n}{2}-1}, g'_{\frac{n}{2}-1} = h_3 g'_{\frac{n}{2}}, g'_{\frac{n}{2}} = h_3 g'_n$ and $g'_n = h_2 g'_1$. So, $g'_1 = h_3^{\frac{n}{2}} g'_n$ and $g'_n = h_2 g'_1$. Therefore, $g'_1 = h_3^{\frac{n}{2}} h_2 g'_1$ and hence $h_3^{\frac{n}{2}} h_2 = 1$. So, $h_2 = h_3^{-\frac{n}{2}}$ and therefore $\langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, that is a contradiction since we know that abelian groups satisfy the zero divisor conjecture. Therefore, with above assumptions on C_1 and C_2 , the graph $K(\alpha, \beta)$ cannot be isomorphic to the graph M_n .

Hence, the statement of this theorem is proven for the case that $T_{C_1} = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$ and $T_{C_2} = [h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$. Since all cycles of length 4 in the graphs L_n and M_n are consecutive, it is easy to see that if we consider $T_{C_1} = [h_2, 1, h_3, h_2, 1, h_2, h_3, 1]$ and $T_{C_2} = [h_2, 1, h_2, h_3, 1, h_2, 1, h_3]$, then the above discussion is true for the latter case too. Therefore, the statement of this theorem is proven for the first item of Remark 4.4.

With a similar discussion such as above, the statement of this theorem can be proven for the other 5 cases in Remark 4.4. Hence, the graph $K(\alpha, \beta)$ is not isomorphic to the graphs L_n and M_n in Figure 8, where the number of vertices of $K(\alpha, \beta)$, L_n and M_n is equal to n . \square

5. Forbidden subgraphs of Kaplansky graphs over \mathbb{F}_2

In previous sections, we studied the existence of triangles and square and some subgraphs containing them in the graph $K_{\mathbb{F}}(\alpha, \beta)$. Also, we saw that K_3 and $K_{2,3}$ are two forbidden subgraphs of the graph $K(\alpha, \beta)$. With the same discussion such as about C_4 cycles, we can study cycles of other lengths in the graph $K(\alpha, \beta)$ by using their relations. In this section, by using cycles up to lengths 7 and their relations, we find another forbidden subgraphs of the graph $K(\alpha, \beta)$. The procedure of finding these forbidden subgraphs is as like as the procedure of finding previous examples. So, the frequent tedious details has been omitted. Such forbidden subgraphs of the Kaplansky graph are listed in Table 1. In the following, we discuss about such subgraphs. Here, forbidden subgraphs are numbered from 1 to 44 such that the forbidden subgraph $K_{2,3}$ is numbered by 1.

5.1. $\mathbf{K_{2,3}}$. By Theorem 2.17, $K(\alpha, \beta)$ contains no subgraph isomorphic to the complete bipartite graph $K_{2,3}$.

5.2. $\mathbf{C_4 - -C_5}$. With the same discussion such as about $K_{2,3}$, it can be seen that there are 121 different cases for the relations of the cycles C_4 and C_5 in this structure. Using GAP [9], we see that the groups with two generators h_2 and h_3 and two relations which are between 111 cases of these 121 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 10 cases for the relations of the cycles C_4 and C_5 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - -C_5$ in $K(\alpha, \beta)$. It can be seen that all of these 10 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - -C_5$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 10 cases is a quotient of $B(1, k)$, for some integer k , or has a torsion element.

5.3. $\mathbf{C_4 - -C_6}$. It can be seen that there are 658 different cases for the relations of the cycles C_4 and C_6 in this structure. By considering all groups with two generators h_2 and h_3 and two relations which are between these cases and by using GAP [9], we see that 632 groups are finite and solvable, or just finite. So, there are just 20 cases for the relations of the cycles C_4 and C_6 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - -C_6$ in $K(\alpha, \beta)$. It can be seen that all of these 20 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - -C_6$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 20 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.4. $\mathbf{C_4 - C_5(-C_5-)}$. It can be seen that there are 42 cases for the relations of a cycle C_4 and two cycles C_5 in the graph $C_4 - C_5(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 38 cases of these 42 cases are finite and solvable, that is a contradiction. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_5-)$ in $K(\alpha, \beta)$. It can be seen that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_5-)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k , or has a torsion element.

5.5. $\mathbf{C_4 - C_5(-C_4-)}$. It can be seen that there are 4 cases for the relations of two cycles C_4 and a cycle C_5 in this structure. By considering all groups with two generators h_2 and h_3 and three relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_4-)$.

5.6. $\mathbf{C_4 - C_5(-C_6--)}$. It can be seen that there are 126 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_6 in the graph $C_4 - C_5(-C_6--)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 122 cases of these 126 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_6--)$ in $K(\alpha, \beta)$. It can be seen that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_6--)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k .

5.7. $\mathbf{C_4 - C_5(-C_6-)}$. It can be seen that there are 462 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_6 in the graph $C_4 - C_5(-C_6-)$. By considering all groups with two generators h_2 and h_3 and two relations which are between these cases and by using GAP [9], we see that 436 groups are finite and solvable, or just finite. So, there are just 22 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_6-)$ in $K(\alpha, \beta)$. It can be shown that all of these 22 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_6-)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 22 cases is a quotient of $B(1, k)$, for some integer k , is a cyclic group or is a solvable group.

5.8. $\mathbf{C}_4 - \mathbf{C}_5(-\mathbf{C}_7 - -)$. It can be seen that there are 648 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_7 in the graph $C_4 - C_5(-C_7 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 608 cases of these 648 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 40 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_7 - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 40 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_7 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 40 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element, is a cyclic group or is a solvable group.

5.9. $\mathbf{C}_5 - -\mathbf{C}_5(- - \mathbf{C}_5)$. It can be seen that there are 192 cases for the relations of three cycles C_5 in the graph $C_5 - -C_5(- - C_5)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 188 cases of these 192 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - -C_5(- - C_5)$ in $K(\alpha, \beta)$. It can be shown that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - -C_5(- - C_5)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k .

5.10. $\mathbf{C}_5 - -\mathbf{C}_5(- - \mathbf{C}_6)$. It can be seen that there are 1006 cases for the relations of two cycles C_5 and a cycle C_6 in the graph $C_5 - -C_5(- - C_6)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 986 cases of these 1006 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 20 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - -C_5(- - C_6)$ in $K(\alpha, \beta)$. It can be shown that all of these 20 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - -C_5(- - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 20 cases is a quotient of $B(1, k)$, for some integer k , is a cyclic group or is a solvable group.

5.11. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(\mathbf{C}_7 - 1)$. It can be seen that there are 176 cases for the relations of a cycle C_4 , two cycles C_7 and a cycle C_6 in this structure. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 1)$.

5.12. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(- - \mathbf{C}_5 -)$. It can be seen that there are 28 cases for the relations of a cycle C_4 , a cycle C_6 , a cycle C_7 and a cycle C_5 in the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 24 cases of these 28 cases are finite and solvable, that is a contradiction with the assumptions. So,

there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$ in $K(\alpha, \beta)$. It can be shown that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a cyclic group.

5.13. $\mathbf{C_4 - C_6(-C_6 - -)(-C_4 -)}$. It can be seen that there is no case for the relations of two cycles C_4 and two cycles C_6 in this structure. It means that the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(-C_4 -)$.

5.14. $\mathbf{C_4 - C_6(-C_6 - -)(- - C_5 -)}$. It can be seen that there are 22 cases for the relations of a cycle C_4 , two cycles C_6 and a cycle C_5 in this structure. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(- - C_5 -)$.

5.15. $\mathbf{C_4 - C_6(-C_6 - -)(C_6 - - -)}$. It can be seen that there are 66 cases for the relations of a cycle C_4 and three cycles C_6 in the graph $C_4 - C_6(-C_6 - -)(C_6 - - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 62 cases of these 66 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(C_6 - - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(C_6 - - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k .

5.16. $\mathbf{C_4 - C_6(-C_6 - -)(- - - C_4)}$. It can be seen that there is no case for the relations of two cycles C_4 and two cycles C_6 in this structure. It means that the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(- - - C_4)$.

5.17. $\mathbf{C_5 - C_5(- - C_6 - -)}$. It can be seen that there are 440 cases for the relations of two cycles C_5 and a cycle C_6 in the graph $C_5 - C_5(- - C_6 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 404 cases of these 440 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 36 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_6 - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 36 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_6 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 36 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.18. $\mathbf{C_5 - C_5(-C_6 - -)(C_6 - -)}$. It can be seen that there are 56 cases for the relations of two cycles C_5 and two cycles C_6 in the graph $C_5 - C_5(-C_6 - -)(C_6 - -)$. By considering all groups with two generators h_2 and h_3 and four relations which are between 54 cases of these 56 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 2 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(C_6 - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 2 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(C_6 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 2 cases is a cyclic group.

5.19. $\mathbf{C_5 - C_5(-C_6 - -)(- - C_6 - 1)}$. It can be seen that there are 56 cases for the relations of two cycles C_5 and two cycles C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 1)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 1)$.

5.20. $\mathbf{C_5 - C_5(-C_6 - -)(-C_5 - -)}$. It can be seen that there are 14 cases for the relations of three cycles C_5 and one cycle C_6 in the graph $C_5 - C_5(-C_6 - -)(-C_5 - -)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(-C_5 - -)$.

5.21. $\mathbf{C_6 - - - C_6(C_6 - - - C_6)}$. It can be seen that there are 46 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 30 cases of these 46 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - - C_6)$ in $K(\alpha, \beta)$. It can be shown that these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 16 cases is a quotient of $B(1, k)$, for some integer k , or has a torsion element.

5.22. $\mathbf{C_6 - - - C_6(C_6)(C_6)(C_6)}$. It can be seen that there are 10 cases for the relations of five C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and five relations which are between 6 cases of these 10 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)(C_6)$ in $K(\alpha, \beta)$. It can be shown that these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)(C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k , or is a cyclic group.

5.23. $\mathbf{C}_5(- - \mathbf{C}_6 - -)\mathbf{C}_5(- - - \mathbf{C}_6)$. It can be seen that there are 134 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 130 cases of these 134 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(- - - C_6)$ in $K(\alpha, \beta)$. It can be shown that these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(- - - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a quotient of $B(1, k)$, for some integer k , or is a cyclic group.

5.24. $\mathbf{C}_6 - - \mathbf{C}_6(\mathbf{C}_6 - - \mathbf{C}_6)$. It can be seen that there are 5119 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 4983 cases of these 5119 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 136 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - C_6(C_6 - - C_6)$ in $K(\alpha, \beta)$. It can be shown that these 136 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - C_6(C_6 - - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 136 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.25. $\mathbf{C}_6 - - - \mathbf{C}_6(\mathbf{C}_6 - - \mathbf{C}_6)$. It can be seen that there are 1594 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 1446 cases of these 1594 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 148 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - C_6)$ in $K(\alpha, \beta)$. It can be shown that these 148 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 148 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.26. $\mathbf{C}_6 - - - \mathbf{C}_6(-\mathbf{C}_5-)$. It can be seen that there are 1482 cases for the relations of two cycles C_6 and a cycle C_5 in the graph $C_6 - - - C_6(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 1358 cases of these 1482 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 124 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(-C_5-)$ in $K(\alpha, \beta)$. It can be shown that all of these 124 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_5-)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 124 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.27. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(- - - \mathbf{C}_6)$. It can be seen that there are 124 cases for the relations of a cycle C_4 , two cycles C_6 and a cycle C_7 in the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$. Using GAP [9], we

see that all groups with two generators h_2 and h_3 and four relations which are between 112 cases of these 124 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 12 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$ in $K(\alpha, \beta)$. It can be shown that all of these 12 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 12 cases is a quotient of $B(1, k)$, for some integer k .

5.28. $\mathbf{C_4 - C_6(- - C_7 - -)(C_4)(C_4)}$. It can be seen that there are 8 cases for the relations of three cycles C_4 , a cycle C_7 and a cycle C_6 in this structure. By considering all groups with two generators h_2 and h_3 and five relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_4)(C_4)$.

5.29. $\mathbf{C_6 - - - C_6(-C_5 - -)}$. It can be seen that there are 418 cases for the relations of two cycles C_6 and a cycle C_5 in the graph $C_6 - - - C_6(-C_5 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 358 cases of these 418 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 60 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(-C_5 - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 60 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_5 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 60 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element, is a cyclic group or is a solvable group.

5.30. $\mathbf{C_6 - - C_6(- - C_5 - -)(-C_5 - -)}$. It can be seen that there are 62 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 56 cases of these 62 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - C_6(- - C_5 - -)(-C_5 - -)$ in $K(\alpha, \beta)$. It can be shown that these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - C_6(- - C_5 - -)(-C_5 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 6 cases has a torsion element or is a cyclic group.

5.31. $\mathbf{C_6 - - C_6(- - C_5 - -)(C_6 - - -)}$. It can be seen that there are 76 cases for the relations of a cycle C_5 and three cycles C_6 in the graph $C_6 - - C_6(- - C_5 - -)(C_6 - - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 64 cases of these 76 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 12 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - C_6(- - C_5 - -)(C_6 - - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 12 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - C_6(- - C_5 - -)(C_6 - - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 12 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.32. $\mathbf{C_5(- - C_6 - -)C_5(C_6)}$. It can be seen that there are 120 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 104 cases of these 120 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_6)$ in $K(\alpha, \beta)$. It can be shown that these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_6)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 16 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.33. $\mathbf{C_5(- - C_6 - -)C_5(C_7)}$. It can be seen that there are 248 cases for the relations of two cycles C_5 , a cycle C_6 and a cycle C_7 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 220 cases of these 248 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 28 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_7)$ in $K(\alpha, \beta)$. It can be shown that these 28 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_7)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 28 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element or is a cyclic group.

5.34. $\mathbf{C_5 - C_5(- - C_7 - -)(- - C_5)}$. It can be seen that there are 394 cases for the relations of a cycle C_7 and three cycles C_5 in the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 352 cases of these 394 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 42 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$ in $K(\alpha, \beta)$. It can be shown that all of these 42 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 42 cases is a quotient of $B(1, k)$, for some integer k , has a torsion element, is a cyclic group or is a solvable group.

5.35. $\mathbf{C_5 - C_5(- - C_7 - -)(-C_5-)}$. It can be seen that there are 138 cases for the relations of a cycle C_7 and three cycles C_5 in the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 132 cases of these 138 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$ in $K(\alpha, \beta)$. It can be shown that all of these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 6 cases has a torsion element, is a cyclic group or is a solvable group.

5.36. $\mathbf{C_5 - C_5(-C_6 - -)(- - C_6 - 2)}$. It can be seen that there are 22 cases for the relations of two cycles C_5 and two cycles C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$.

5.37. $\mathbf{C_4 - C_4(-C_7 - -)(C_4)}$. It can be seen that there are 32 cases for the relations of a cycle C_7 and three cycles C_4 in the graph $C_4 - C_4(-C_7 - -)(C_4)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_4(-C_7 - -)(C_4)$.

5.38. $\mathbf{C_5 - -C_5(-C_5 - -)}$. It can be seen that there are 64 cases for the relations of three cycles C_5 in the graph $C_5 - -C_5(-C_5 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 58 cases of these 64 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - -C_5(-C_5 - -)$ in $K(\alpha, \beta)$. It can be shown that all of these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - -C_5(-C_5 - -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 6 cases is a quotient of $B(1, k)$, for some integer k or is a cyclic group.

5.39. $\mathbf{C_6 - - - C_6(-C_4)}$. It can be seen that there are 420 cases for the relations of two cycles C_6 and a cycle C_4 in the graph $C_6 - - - C_6(-C_4)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 398 cases of these 420 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 22 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(-C_4)$ in $K(\alpha, \beta)$. It can be shown that all of these 22 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_4)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 22 cases is a cyclic group.

5.40. $\mathbf{C_6 - -C_6(C_4)}$. It can be seen that there are 279 cases for the relations of two cycles C_6 and a cycle C_4 in the graph $C_6 - -C_6(C_4)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 268 cases of these 279 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 11 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(C_4)$ in $K(\alpha, \beta)$. It can be shown that all of these 11 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - -C_6(C_4)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 11 cases is a quotient of $B(1, k)$, for some integer k , or is a cyclic group.

5.41. $\mathbf{C}_4 - \mathbf{C}_6(-\mathbf{C}_4)(-\mathbf{C}_4)$. It can be seen that there are 36 cases for the relations of a cycle C_6 and three cycles C_4 in the graph $C_4 - C_6(-C_4)(-C_4)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_4)(-C_4)$.

5.42. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(-\mathbf{C}_5 -)$. It can be seen that there are 62 cases for the relations of a cycle C_4 , a cycle C_6 , a cycle C_7 and a cycle C_5 in the graph $C_4 - C_6(- - C_7 - -)(-C_5 -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 58 cases of these 62 cases are solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(-C_5 -)$ in $K(\alpha, \beta)$. It can be shown that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(-C_5 -)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 4 cases is a cyclic group.

5.43. $\mathbf{C}_5 - \mathbf{C}_5(-\mathbf{C}_6 - -)(- - \mathbf{C}_5 -)$. It can be seen that there are 14 cases for the relations of three cycles C_5 and a cycle C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_5 -)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_5 -)$.

5.44. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(\mathbf{C}_7 - 2)$. It can be seen that there are 168 cases for the relations of a cycle C_4 , a cycle C_6 and two cycles C_7 in the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 152 cases of these 168 cases are solvable, that is a contradiction with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$ in $K(\alpha, \beta)$. It can be shown that all of these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$.

Each group with two generators h_2 and h_3 and two relations which are between the latter 16 cases is a quotient of $B(1, k)$, for some integer k , or is a cyclic group.

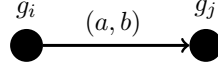
In the following theorem, we summarize our results about forbidden subgraphs of the graph $K(\alpha, \beta)$.

Theorem 5.1. *The Kaplansky graph $K(\alpha, \beta)$ is a triangle-free graph which contains no subgraph isomorphic to one of the 46 other graphs in Table 1.*

Remark 5.2. In $K_{\mathbb{F}}(\alpha, \beta)$, for all $g_i, g_j \in \text{supp}(\beta)$, $g_i \sim g_j$ if and only if $ag_i = bg_j$ for some $a, b \in \text{supp}(\alpha)$. Let e be the edge between such latter vertices. Suppose that we give an orientation and a label in $L = \{(1, h_2), (1, h_3), (h_2, h_3)\}$ to e such as follows:

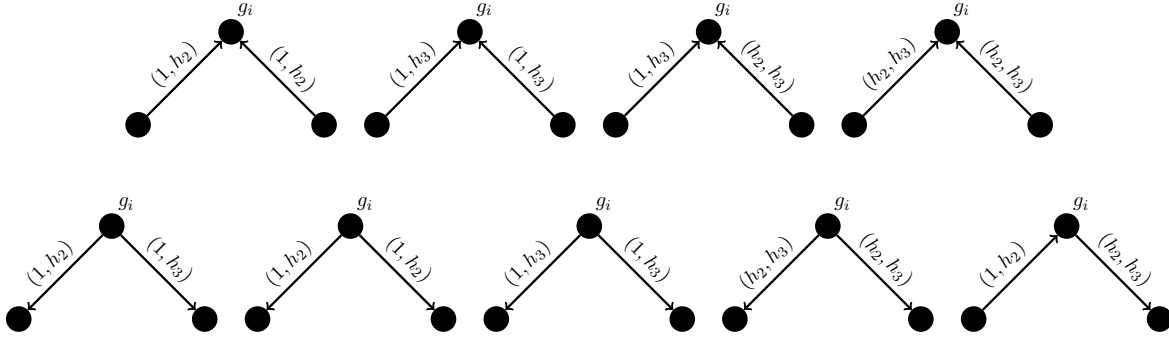
- (1) If $(a, b) \in L$, then e is labelled by (a, b) and oriented from g_i to g_j .
- (2) If $(b, a) \in L$, then e is labelled by (b, a) and oriented from g_j to g_i .

Therefore, $K_{\mathbb{F}}(\alpha, \beta)$ can be considered as a directed graph in which all edges are labelled by L with the above method.

FIGURE 10. An edge in $K_{\mathbb{F}}(\alpha, \beta)$

Let e be an edge in $K_{\mathbb{F}}(\alpha, \beta)$ as Figure 10, where $(a, b) \in L$. Suppose that when we traverse e from g_i to g_j (or from g_j to g_i), the word $a^{-1}b$ (or $b^{-1}a$, respectively) is corresponded to this edge. Then G is the group $\langle h_2, h_3 \mid \text{words of closed paths in } K_{\mathbb{F}}(\alpha, \beta) \rangle$.

Let $\mathbb{F} = \mathbb{F}_2$. It is easy to see that for each vertex g_i of $K_{\mathbb{F}_2}(\alpha, \beta)$, none of the cases in Figure 11 can be happened.

FIGURE 11. Impossible cases for labelling the edges of each vertex g_i in $K_{\mathbb{F}_2}(\alpha, \beta)$

Remark 5.3. Suppose that Γ is a directed graph in which all edges are labelled by L . Let e be an edge in Γ as Figure 10, where $(a, b) \in L$. Suppose that when we traverse e from g_i to g_j (or from g_j to g_i), the word $a^{-1}b$ (or $b^{-1}a$, respectively) is corresponded to this edge. Let $G(\Gamma) := \langle h_2, h_3 \mid \text{words of closed paths in } \Gamma \rangle$. Suppose that the ground graph of Γ contains a subgraph isomorphic to one of the graphs in Table 1 and for each vertex g_i of such latter subgraph, none of the cases in Figure 11 are happened. By Theorem 5.1 and Remark 5.2, $G(\Gamma)$ has at least one of the following properties:

- (1) It is a finite group,
- (2) It is an abelian group,
- (3) It is a quotient of $BS(1, k)$ or $BS(k, 1)$, where k is an integer,
- (4) It has a non-trivial torsion element,
- (5) It is a solvable group.

6. The possible number of vertices of $K(\alpha, \beta)$

By Remark 4.2, the number of vertices of $K(\alpha, \beta)$ must be an even positive integer $n \geq 4$. Also by Theorem 2.9, the graph $K(\alpha, \beta)$ is a connected cubic triangle-free graph and we found 44 other forbidden subgraphs of such graph in Section 5. Furthermore in Section 4, we found two graphs, namely L_n and M_n , with n vertices which are not isomorphic to the graph $K(\alpha, \beta)$.

Using Sage Mathematics Software [18] and its package *Nauty-geng*, all non-isomorphic connected cubic triangle-free graphs with the size of the vertex sets n can be found. In this section by using

Sage Mathematics Software [18], we give some results about checking each of the mentioned forbidden subgraphs in all of the non-isomorphic connected cubic triangle-free graphs with the size of vertex sets $n \leq 20$. By using these results we show that n must be greater than or equal to 20. Also, some results about the case $n = 20$ is given.

Table 3 lists all results about the number of non-isomorphic connected cubic triangle-free graphs with the size of vertex sets $n \leq 20$ which contain each of the forbidden subgraphs. The results in this table from top to bottom are presented in such a way that by checking each of the forbidden subgraphs in a row, the number of graphs containing these subgraph are omitted from the total number and the existence of the next forbidden subgraph is checked among the remaining ones.

Table 3: Existence of the forbidden subgraphs in non-isomorphic connected cubic triangle-free graphs with the size of vertex sets $n \leq 20$

	$n = 4$	$n = 6$	$n = 8$	$n = 10$	$n = 12$	$n = 14$	$n = 16$	$n = 18$	$n = 20$
Total	0	1	2	6	22	110	792	7805	97546
1) $K_{2,3}$	0	1	0	1	4	22	144	1222	12991
2) $C_4 - -C_5$	0	0	1	2	6	30	223	2161	25427
3) $C_4 - -C_6$	0	0	1	1	6	31	223	2228	28080
4) $C_4 - C_5(-C_5-)$	0	0	0	0	2	6	40	319	3396
5) $C_4 - C_5(-C_4-)$	0	0	0	1	0	3	12	88	1123
6) $C_4 - C_5(-C_6--)$	0	0	0	0	0	4	42	389	4548
7) $C_4 - C_5(-C_6-)$	0	0	0	0	0	1	20	382	5661
8) $C_4 - C_5(-C_7--)$	0	0	0	0	0	1	10	176	3172
9) $C_5--C_5(-C_5)$	0	0	0	1	2	3	18	157	1617
10) $C_5--C_5(-C_6)$	0	0	0	0	0	4	32	291	4289
11) $C_4 - C_6(-C_7--)(C_7-1)$	0	0	0	0	1	0	9	64	446
12) $C_4 - C_6(-C_7--)(-C_5-)$	0	0	0	0	0	0	0	2	51
13) $C_4 - C_6(-C_6--)(-C_4-)$	0	0	0	0	1	0	1	5	35
14) $C_4 - C_6(-C_6--)(-C_5-)$	0	0	0	0	0	0	1	1	149
15) $C_4 - C_6(-C_6--)(C_6---)$	0	0	0	0	0	0	1	30	404
16) $C_4 - C_6(-C_6--)(---C_4)$	0	0	0	0	0	0	1	4	12
17) $C_5 - C_5(-C_6--)$	0	0	0	0	0	2	0	41	352
18) $C_5 - C_5(-C_6--)(C_6---)$	0	0	0	0	0	0	7	47	529
19) $C_5 - C_5(-C_6--)(-C_6-1)$	0	0	0	0	0	0	1	31	249
20) $C_5 - C_5(-C_6--)(-C_5--)$	0	0	0	0	0	0	1	1	69
21) $C_6---C_6(C_6---C_6)$	0	0	0	0	0	1	1	8	43
22) $C_6---C_6(C_6)(C_6)(C_6)$	0	0	0	0	0	0	2	6	25
23) $C_5(-C_6--)C_5(---C_6)$	0	0	0	0	0	0	1	16	374
24) $C_6--C_6(C_6--C_6)$	0	0	0	0	0	0	0	29	438
25) $C_6---C_6(C_6--C_6)$	0	0	0	0	0	0	0	20	505
26) $C_6---C_6(-C_5-)$	0	0	0	0	0	0	0	27	721
27) $C_4 - C_6(-C_7--)(---C_6)$	0	0	0	0	0	0	0	8	257
28) $C_4 - C_6(-C_7--)(C_4)(C_4)$	0	0	0	0	0	0	0	2	2
29) $C_6---C_6(-C_5--)$	0	0	0	0	0	0	0	5	66
30) $C_6--C_6(-C_5-)(-C_5-)$	0	0	0	0	0	0	0	12	293
31) $C_6--C_6(-C_5-)(C_6---)$	0	0	0	0	0	0	0	2	267
32) $C_5(-C_6--)C_5(C_6)$	0	0	0	0	0	0	0	1	43
33) $C_5(-C_6--)C_5(C_7)$	0	0	0	0	0	0	0	2	50

Continued on next page

Table 3 – continued from previous page

	$n = 4$	$n = 6$	$n = 8$	$n = 10$	$n = 12$	$n = 14$	$n = 16$	$n = 18$	$n = 20$
34) $C_5 - C_5(- - C_7 - -)(- - C_5)$	0	0	0	0	0	0	0	6	199
35) $C_5 - C_5(- - C_7 - -)(- C_5 -)$	0	0	0	0	0	0	0	2	69
36) $C_5 - C_5(- C_6 - -)(- - C_6 - 2)$	0	0	0	0	0	0	0	2	114
37) $C_4 - C_4(- C_7 -)(C_4)$	0	0	0	0	0	1	0	6	72
38) $C_5 - - C_5(- C_5 - -)$	0	0	0	0	0	0	0	1	11
39) $C_6 - - - C_6(- C_4)$	0	0	0	0	0	0	0	3	94
40) $C_6 - - C_6(C_4)$	0	0	0	0	0	0	0	2	67
41) $C_4 - C_6(- C_4)(- C_4)$	0	0	0	0	0	0	0	1	30
42) $C_4 - C_6(- - C_7 - -)(- C_5 -)$	0	0	0	0	0	0	0	1	26
43) $C_5 - C_5(- C_6 - -)(- - C_5 -)$	0	0	0	0	0	0	0	1	17
44) $C_4 - C_6(- - C_7 - -)(C_7 - 2)$	0	0	0	0	0	0	0	1	41
Isomorphic to L_n	0	0	0	0	0	0	1	1	1
Isomorphic to M_n	0	0	0	0	0	1	1	1	1
Remains	0	0	0	0	0	0	0	0	1120

The discussion above and the results of Table 3 are summarized in the following theorem.

Theorem 6.1. *The vertex set size of $K(\alpha, \beta)$ must be greater than or equal to 20. Furthermore, there are just 1120 graphs with vertex set size equal to 20 which may be isomorphic to $K(\alpha, \beta)$.*

Corollary 6.2. *Let α and β be non-zero elements of the group algebra of any torsion-free group over the field with two elements. If $|supp(\alpha)| = 3$ and $\alpha\beta = 0$ then $|supp(\beta)| \geq 20$.*

7. Some results on the unit conjecture

Throughout this section let \mathbb{F} be an arbitrary field and G be a torsion-free group and $\gamma = \gamma_1 h_1 + \gamma_2 h_2 + \gamma_3 h_3 \in \mathbb{F}[G]$ such that $|supp(\gamma)| = 3$. Suppose further that $\gamma\delta = 1$ for some $\delta \in \mathbb{F}[G]$ and assume that $n := |supp(\delta)|$ is minimum with respect to the latter property. Let $\delta = \delta_1 g_1 + \delta_2 g_2 + \dots + \delta_n g_n$ and $A = \{1, 2, 3\} \times \{1, 2, \dots, n\}$. Since $\gamma\delta = 1$, there must be at least one $(i, j) \in A$ such that $h_i g_j = 1$. By renumbering, we may assume that $(i, j) = (1, 1)$. Replacing γ by $h_1^{-1} \gamma$ and δ by δg_1^{-1} we may assume that $h_1 = g_1 = 1$. So we may suppose that $1 \in supp(\gamma)$ and $1 \in supp(\delta)$.

There is a partition π of A such that (i, j) and (i', j') belong to the same set of π if and only if $h_i g_j = h_{i'} g_{j'}$ and because of the relation $\gamma\delta = 1$, for all $E \in \pi$ we have

$$(7.1) \quad \sum_{(i,j) \in E} \gamma_i \delta_j = \begin{cases} 1 & (1, 1) \in E \\ 0 & (1, 1) \notin E \end{cases}$$

Let E_1 be the set in π which contains $(1, 1)$. Obviously, $n \geq 2$. Abelian groups satisfy Conjecture 1.2. So, G must be a nonabelian torsion-free group. Firstly in this section we show that $n \geq 4$. Then, we examine some small positive integers greater than 3 as the possible values of n and show that n must be at least 8.

7.1. The support of δ is of size at least 4.

Theorem 7.1 (Corollary of [12]). *Let G be an arbitrary group and let B and C be finite non-empty subsets of G . Suppose that each non-identity element g of G has a finite or infinite order greater than or equal to $|B| + |C| - 1$. Then $|BC| \geq |B| + |C| - 1$.*

By Theorem 7.1, $|supp(\gamma)supp(\delta)| \geq |supp(\gamma)| + |supp(\delta)| - 1$ because G is torsion-free. Hence, $3n \geq |supp(\gamma)supp(\delta)| \geq 2 + n$.

- (1) Let $n = 2$. Then with the above discussion, $6 \geq |supp(\gamma)supp(\delta)| \geq 4$ and so, there are at least 3 sets different from E_1 in π , namely E_2, E_3, E_4 . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, 4\}$, but since $6 - 4 = 2$, $|E_k| \leq 1$ for some $k \in \{2, 3, 4\}$, a contradiction. Therefore, $n \neq 2$.
- (2) Let $n = 3$. Then with the above discussion, $9 \geq |supp(\gamma)supp(\delta)| \geq 5$ and so, there are at least 4 sets different from E_1 in π , namely E_2, E_3, E_4, E_5 . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, 4, 5\}$, but since $9 - 5 = 4$, $|E_1| = 1$ and $|E_k| = 2$ for all $k \in \{2, 3, 4, 5\}$. Therefore $E_1 = \{(1, 1)\}$ and for all $k \in \{2, 3, 4, 5\}$, $E_k = \{(i, j), (i', j')\}$ where $h_i g_j = h_{i'} g_{j'}$ for some $(i, j), (i', j') \in A$ such that $i \neq i'$ and $j \neq j'$. Let $\gamma' = \sum_{a \in supp(\gamma)} a$ and $\delta' = \sum_{b \in supp(\delta)} b$. So, $\gamma', \delta' \in \mathbb{F}_2[G]$, $|supp(\gamma')| = 3$ and $|supp(\delta')| = 3$ and with the above discussion we have $\gamma'\delta' = 1$, that is a contradiction with Theorem 3.2. Therefore, $n \neq 3$.

7.2. The support of δ must be of size greater than or equal to 8. Without loss of generality we may assume that G is generated by $supp(\gamma) \cup supp(\delta)$, since otherwise we replace G by the subgroup generated by this set. Also, $1 \in supp(\gamma)$ and $1 \in supp(\delta)$ and with the same discussion such as in Lemma 2.1, $\langle supp(\delta) \rangle = \langle supp(\gamma) \rangle$. Therefore by Theorem 3.1, if $n \geq 4$, then $|supp(\gamma)supp(\delta)| \geq |supp(\gamma)| + |supp(\delta)| + 1$. Also, it is easy to see that $|supp(\gamma)||supp(\delta)| \geq |supp(\gamma)supp(\delta)|$. Hence, $3n \geq |supp(\gamma)supp(\delta)| \geq 4 + n$.

- (1) Let $n = 4$. Then with the above discussion, $12 \geq |supp(\gamma)supp(\delta)| \geq 8$ and so, there are at least 7 sets different from E_1 in π , namely E_2, E_3, \dots, E_8 . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, \dots, 8\}$, but since $12 - 8 = 4$, $|E_k| \leq 1$ for some $k \in \{2, 3, \dots, 8\}$, a contradiction. Therefore, $n \neq 4$.
- (2) Let $n = 5$. Then with the discussion above $15 \geq |supp(\gamma)supp(\delta)| \geq 9$ and so, there are at least 8 sets different from E_1 in π , namely E_2, E_3, \dots, E_9 . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, \dots, 9\}$, but since $15 - 9 = 6$, $|E_k| \leq 1$ for some $k \in \{2, 3, \dots, 9\}$, a contradiction. Therefore, $n \neq 5$.
- (3) Let $n = 6$. Then with the discussion above $18 \geq |supp(\gamma)supp(\delta)| \geq 10$ and so, there are at least 9 sets different from E_1 in π , namely E_2, E_3, \dots, E_{10} . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, \dots, 10\}$, but since $18 - 10 = 8$, $|E_k| \leq 1$ for some $k \in \{2, 3, \dots, 10\}$, a contradiction. Therefore, $n \neq 6$.
- (4) Let $n = 7$. Then with the discussion above $21 \geq |supp(\gamma)supp(\delta)| \geq 11$ and so, there are at least 10 sets different from E_1 in π , namely E_2, E_3, \dots, E_{11} . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, \dots, 11\}$, but since $21 - 11 = 10$, $|E_1| = 1$ and $|E_k| = 2$ for all $k \in \{2, 3, \dots, 11\}$. Therefore $E_1 = \{(1, 1)\}$ and for all

$k \in \{2, 3, \dots, 11\}$, $E_k = \{(i, j), (i', j')\}$ where $h_i g_j = h_{i'} g_{j'}$ for some $(i, j), (i', j') \in A$ such that $i \neq i'$ and $j \neq j'$. Let $\gamma' = \sum_{a \in \text{supp}(\gamma)} a$ and $\delta' = \sum_{b \in \text{supp}(\delta)} b$. So, $\gamma', \delta' \in \mathbb{F}_2[G]$, $|\text{supp}(\gamma')| = 3$ and $|\text{supp}(\delta')| = 7$ and with the above discussion we have $\gamma'\delta' = 1$, that is a contradiction with Theorem 3.2. Therefore, $n \neq 7$.

(5) Let $n = 8$. Then with the discussion above $24 \geq |\text{supp}(\gamma)\text{supp}(\delta)| \geq 12$. Let $|\text{supp}(\gamma)\text{supp}(\delta)| > 12$. Then $|\text{supp}(\gamma)\text{supp}(\delta)| \geq 13$ and so, there are at least 12 sets different from E_1 in π , namely E_2, E_3, \dots, E_{13} . Since $\gamma\delta = 1$, each of such sets must have at least two elements such that $\sum_{(i,j) \in E_k} \gamma_i \delta_j = 0$ for all $k \in \{2, 3, \dots, 13\}$, but since $24 - 13 = 11$, $|E_k| \leq 1$ for some $k \in \{2, 3, \dots, 13\}$, a contradiction. So, $|\text{supp}(\gamma)\text{supp}(\delta)| = 12$. Therefore, there are 11 sets different from E_1 in π , namely E_2, E_3, \dots, E_{12} . Since $\gamma\delta = 1$, there are two cases for the number of elements in such sets.

- (a) $|E_1| = 2$ and for all $k \in \{2, 3, \dots, 12\}$, $E_k = \{(i, j), (i', j')\}$ where $h_i g_j = h_{i'} g_{j'}$ for some $(i, j), (i', j') \in A$ such that $i \neq i'$ and $j \neq j'$. Let $\gamma' = \sum_{a \in \text{supp}(\gamma)} a$ and $\delta' = \sum_{b \in \text{supp}(\delta)} b$. So, $\gamma', \delta' \in \mathbb{F}_2[G]$, $|\text{supp}(\gamma')| = 3$ and $|\text{supp}(\delta')| = 8$ and with the above discussion we have $\gamma'\delta' = 0$, that is a contradiction (see Corollary 6.2).
- (b) $E_1 = \{(1, 1)\}$ and $|E_l| = 3$ for exactly one $l \in \{2, 3, \dots, 12\}$ and for all $k \in \{2, 3, \dots, 12\} \setminus \{l\}$, $E_k = \{(i, j), (i', j')\}$ where $h_i g_j = h_{i'} g_{j'}$ for some $(i, j), (i', j') \in A$ such that $i \neq i'$ and $j \neq j'$.

Theorem 7.2. *Let γ and δ be elements of the group algebra of any torsion-free group over an arbitrary field. If $|\text{supp}(\gamma)| = 3$ and $\gamma\delta = 1$ then $|\text{supp}(\delta)| \geq 8$.*

7.3. Kaplansky unit graphs over \mathbb{F} . By the discussion from the beginning of the section, similar to the case of zero divisors, it can be associated a graph to γ and δ with the vertex set $\text{supp}(\delta)$ such that two vertices g_i and g_j are adjacent whenever $h_{i'} g_i = h_{j'} g_j$ for some distinct $i', j' \in \{1, 2, 3\}$. We call the graph Kaplansky unit graph of (γ, δ) over \mathbb{F} and it is denoted by $Ku_{\mathbb{F}}(\gamma, \delta)$. The connectedness follows from the way we have chosen δ of minimum support size with respect to the property $\gamma\delta = 1$. Also by Lemma 2.3, $|S| = 6$ where $S = \{h^{-1}h' \mid h, h' \in \text{supp}(\gamma), h \neq h'\}$. Therefore, $Ku_{\mathbb{F}}(\gamma, \delta)$ is a simple graph. Furthermore by Theorems 2.11 and 2.17, with a similar discussion as about Kaplansky graphs, $Ku_{\mathbb{F}}(\gamma, \delta)$ contains no $K_3 - K_3$ or $K_{2,3}$ as a subgraph, too.

By item (5) of Subsection 7.2, if $n = 8$, $|\text{supp}(\gamma)\text{supp}(\delta)| = 12$ and there are exactly 11 sets different from E_1 in π , namely E_2, E_3, \dots, E_{12} . Also, $E_1 = \{(1, 1)\}$ and $|E_l| = 3$ for exactly one $l \in \{2, 3, \dots, 12\}$ and for all $k \in \{2, 3, \dots, 12\} \setminus \{l\}$, $E_k = \{(i, j), (i', j')\}$ such that $i \neq i'$, $j \neq j'$ and $h_i g_j = h_{i'} g_{j'}$.

Let $E_l = \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$. Therefore, $h_{i_1} g_{j_1} = h_{i_2} g_{j_2} = h_{i_3} g_{j_3}$ and so there is a triangle in $Ku_{\mathbb{F}}(\gamma, \delta)$ with the vertex set $\{g_{j_1}, g_{j_2}, g_{j_3}\}$ and there is no other triangle in the latter graph. Let $(2, 1), (3, 1) \notin E_l$. Then by the way we have chosen E_k for $k \in \{1, 2, 3, \dots, 12\}$, the degree of g_{j_1} , g_{j_2} and g_{j_3} are equal to 4, denoted by $\deg(g_{j_1}) = \deg(g_{j_2}) = \deg(g_{j_3}) = 4$. So, there must be 6 other vertices different from g_{j_1} , g_{j_2} and g_{j_3} in the vertex set of $Ku_{\mathbb{F}}(\gamma, \delta)$ because there is no other triangle in the latter graph. This leads us to a contradiction because the size of the vertex set of $Ku_{\mathbb{F}}(\gamma, \delta)$ is $n = 8$. Hence, $E_l = \{(a, 1), (i, j), (i', j')\}$ where $a \in \{2, 3\}$ and $\{h_a, h_i, h_{i'}\} = \text{supp}(\gamma)$. Since $|E_1| = 1$, $1 = h_1 g_1 \neq h_m g_n$ for all $(m, n) \in A \setminus E_1$. So, $\deg(g_1) = 3$ and by renumbering, we may assume that

$Ku_{\mathbb{F}}(\gamma, \delta)$ has the graph H in Figure 12 as a subgraph and there is no other vertices in $Ku_{\mathbb{F}}(\gamma, \delta)$. In H , $g_5 \sim g_2$ or $g_5 \not\sim g_2$.

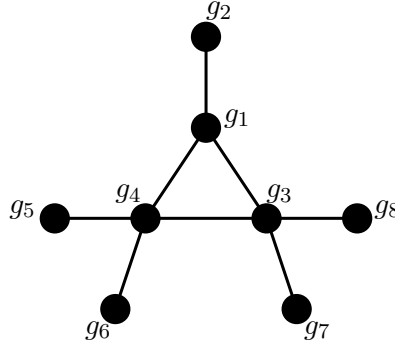


FIGURE 12. The subgraph H of $Ku_{\mathbb{F}}(\gamma, \delta)$ for the case that $n = 8$

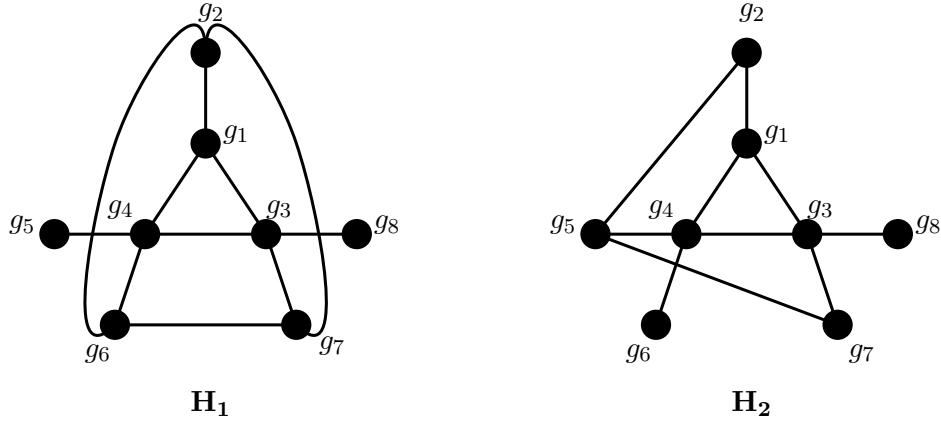


FIGURE 13. Two possible subgraphs of $Ku_{\mathbb{F}}(\gamma, \delta)$ for the case that $n = 8$

Let $g_5 \not\sim g_2$. Since $(a, 2), (a, 5), (a, 6), (a, 7), (a, 8) \notin E_l, E_1$ for all $a \in \{1, 2, 3\}$ and $|E_k| = 2$ for all $k \in \{2, 3, \dots, 12\} \setminus \{l\}$, $\deg(g_2) = \deg(g_5) = \deg(g_6) = \deg(g_7) = \deg(g_8) = 3$. Since $Ku_{\mathbb{F}}(\gamma, \delta)$ is a simple graph which contains exactly one triangle, $g_6 \not\sim g_i$ for all $i \in \{1, 3, 4, 5, 6\}$. So, without loss of generality we may assume that $g_6 \sim g_7$ since $\deg(g_6) = 3$. Suppose, for a contradiction, that $g_6 \sim g_8$. Then there is a subgraph isomorphic to $K_{2,3}$ in $Ku_{\mathbb{F}}(\gamma, \delta)$ with the vertex set $\{g_3, g_4, g_6, g_7, g_8\}$, a contradiction. So, $g_6 \sim g_2$. With a same discussion we have $g_7 \sim g_2$ and $Ku_{\mathbb{F}}(\gamma, \delta)$ has the graph H_1 in Figure 13 as a subgraph. Since $\deg(g_2) = \deg(g_5) = \deg(g_6) = \deg(g_7) = \deg(g_8) = 3$ and $\deg(g_1) = \deg(g_3) = \deg(g_4) = 4$, we must have $g_5 \sim g_8$ with a double edge, a contradiction because $Ku_{\mathbb{F}}(\gamma, \delta)$ is simple. Therefore, $g_5 \sim g_2$.

Since $Ku_{\mathbb{F}}(\gamma, \delta)$ is a simple graph which contains exactly one triangle, $g_5 \not\sim g_i$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. So, $g_5 \sim g_7$ or $g_5 \sim g_8$ because $\deg(g_5) = 3$. As we can see in Figure 12, without loss of generality we may assume that $g_5 \sim g_7$ and $Ku_{\mathbb{F}}(\gamma, \delta)$ has the graph H_2 in Figure 13 as a subgraph. Since $Ku_{\mathbb{F}}(\gamma, \delta)$ is a simple graph which contains exactly one triangle, $g_6 \not\sim g_i$ for all $i \in \{1, 3, 4, 5, 6\}$. So, $g_6 \sim g_2$, $g_6 \sim g_7$ or $g_6 \sim g_8$ because $\deg(g_6) = 3$. If $g_6 \sim g_2$ or $g_6 \sim g_7$ then there is a subgraph isomorphic to

$K_{2,3}$ in $Ku_{\mathbb{F}}(\gamma, \delta)$ as we can see in the graph H_2 of Figure 13, a contradiction. Therefore we must have $g_6 \sim g_8$ with a double edge, a contradiction because $Ku_{\mathbb{F}}(\gamma, \delta)$ is simple.

Hence with the above discussion, $n \neq 8$ and by Theorem 7.2, we have the following result.

Theorem 7.3. *Let γ and δ be elements of the group algebra of any torsion-free group over an arbitrary field. If $|supp(\gamma)| = 3$ and $\gamma\delta = 1$ then $|supp(\delta)| \geq 9$.*

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8. Appendix

In Section 5, we found 44 forbidden subgraphs for the Kaplansky graphs over \mathbb{F}_2 without details. In this section, we give more details about finding such subgraph of $K(\alpha, \beta)$. Firstly with the same discussion such as about C_3 and C_4 cycles, we study cycles of lengths 5 and 6 with their relations. Then we use such relations and the relations of C_4 and C_7 cycles to show that Kaplansky graphs do not contain the latter 44 graphs. Such forbidden subgraphs are numbered from 1 to 44 such that the forbidden subgraph $K_{2,3}$ is numbered by 1.

C_5 cycles: With the same discussion such as about C_4 cycles, there are 105 non-equivalent cases for the relations of a C_5 cycle in the graph $K(\alpha, \beta)$. Such relations are listed in table 4. In the following, we show that some of these relations lead to a contradiction. Each of such relations is marked by a * in the Table 4.

- (1) $h_2^5 = 1$:
 $h_2^5 = 1$ and G is torsion-free $\Rightarrow h_2 = 1$, a contradiction.
- (2) $h_2^4 h_3 = 1$:
 $h_3 = h_2^{-4} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (3) $h_2^4 h_3^{-1} h_2 = 1$:
 $h_3 = h_2^5 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $h_2^3 h_3^{-1} h_2^{-1} h_3 = 1$:
 $h_3^{-1} h_2 h_3 = h_2^3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3)$ is solvable, a contradiction.
- (8) $h_2^3 h_3^{-1} h_2 h_3 = 1$:
 $h_3^{-1} h_2 h_3 = h_2^{-3} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3)$ is solvable, a contradiction.
- (10) $h_2(h_2 h_3)^2 = 1$:
 $h_2 = (h_2 h_3)^{-2}$ and $\langle h_2, h_3 \rangle = \langle h_2, h_2 h_3 \rangle \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 h_3 \rangle$ is abelian, a contradiction.
- (11) $h_2^2 h_3 h_2 h_3^{-1} h_2 = 1$:
 $h_3^{-1} h_2^{-3} h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (15) $h_2^2 h_3 h_2^{-1} h_3^{-1} h_2 = 1$:
 $h_3^{-1} h_2^3 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (27) $(h_2^2 h_3^{-1})^2 h_2 = 1$:
 $h_2 = (h_3 h_2^{-2})^2$ and $\langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-2} \rangle \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-2} \rangle$ is abelian, a contradiction.
- (34) $(h_2 h_3)^2 h_3 = 1$:
 By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.
- (36) $h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 Using Tietze transformation where $h_2 \mapsto h_2 h_3^{-1}$ and $h_3 \mapsto h_3$, we have:
 $h_2^2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (41) $h_2 h_3^4 = 1$:
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

Table 4: All non-equivalent cases for the relations of a C_5 cycle

n	R	n	R	n	R
1	$h_2^5 = 1 *$	36	$h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1 *$	71	$h_2 h_3^{-3} h_2 h_3 = 1$
2	$h_2^4 h_3 = 1 *$	37	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	72	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
3	$h_2^4 h_3^{-1} h_2 = 1 *$	38	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	73	$h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 = 1 *$
4	$h_2^3 h_3^2 = 1$	39	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	74	$h_2 h_3^{-2} h_2 h_3^2 = 1$
5	$h_2^3 h_3 h_2^{-1} h_3 = 1$	40	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$	75	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$
6	$h_2^3 h_3^{-1} h_2^{-1} h_3 = 1 *$	41	$h_2 h_3^4 = 1 *$	76	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1 *$
7	$h_2^3 h_3^{-2} h_2 = 1$	42	$h_2 h_3^3 h_2^{-1} h_3 = 1 *$	77	$(h_2 h_3^{-2})^2 h_2 = 1 *$
8	$h_2^3 h_3^{-1} h_2 h_3 = 1 *$	43	$h_2 h_3^2 h_2^{-2} h_3 = 1$	78	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
9	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	44	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	79	$h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1 *$
10	$h_2 (h_2 h_3)^2 = 1 *$	45	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	80	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$
11	$h_2^2 h_3 h_2 h_3^{-1} h_2 = 1 *$	46	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	81	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 = 1 *$
12	$h_2^2 h_3^3 = 1$	47	$h_2 h_3 h_2^{-2} h_3^2 = 1$	82	$h_2 h_3^{-1} h_2 h_3^3 = 1$
13	$h_2^2 h_3^2 h_2^{-1} h_3 = 1$	48	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	83	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$
14	$h_2^2 h_3 h_2^{-2} h_3 = 1$	49	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	84	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
15	$h_2^2 h_3 h_2^{-1} h_3^{-1} h_2 = 1 *$	50	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	85	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
16	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	51	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1 *$	86	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
17	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	52	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	87	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
18	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	53	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	88	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
19	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	54	$h_2 h_3 h_2^{-1} h_3^3 = 1 *$	89	$(h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1$
20	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	55	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	90	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
21	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	56	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	91	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
22	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	57	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	92	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
23	$h_2^2 h_3^{-3} h_2 = 1$	58	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	93	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
24	$h_2^2 h_3^{-2} h_2 h_3 = 1$	59	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	94	$(h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1 *$
25	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$	60	$h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1 *$	95	$(h_2 h_3^{-1})^3 h_3^{-1} h_2 = 1 *$
26	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	61	$h_2 h_3^{-1} h_2^{-1} h_3^3 = 1 *$	96	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
27	$(h_2^2 h_3^{-1})^2 h_2 = 1 *$	62	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1 *$	97	$(h_2 h_3^{-1})^4 h_2 = 1 *$
28	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	63	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1 *$	98	$h_3^5 = 1 *$
29	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	64	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	99	$h_3^4 h_2^{-1} h_3 = 1 *$
30	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	65	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	100	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
31	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	66	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 = 1 *$	101	$(h_3^2 h_2^{-1})^2 h_3 = 1 *$
32	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	67	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	102	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$
33	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	68	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	103	$(h_3 h_2^{-1} h_3)^2 h_2^{-1} h_3 = 1 *$
34	$(h_2 h_3)^2 h_3 = 1 *$	69	$h_2 h_3^{-3} h_2^{-1} h_3 = 1 *$	104	$(h_3 h_2^{-1})^4 h_3 = 1 *$
35	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	70	$h_2 h_3^{-4} h_2 = 1$	105	$(h_2^{-1} h_3)^5 = 1 *$

$$(42) \quad h_2 h_3^3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(51) \quad h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(54) \quad h_2 h_3 h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(60) \quad h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$.

Using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$$h_3^{-2} h_2^{-1} h_3 h_2 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2) \text{ is solvable, a contradiction.}$$

$$(61) \quad h_2 h_3^{-1} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction

$$(62) \quad h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have $h_3 h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} = 1$.

Using Tietze transformation again where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have:

$$h_3 h_2 h_3^{-1} h_2^{-2} = 1 \Rightarrow h_3^{-1} h_2^2 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1) \text{ is solvable, a contradiction.}$$

$$(63) \quad h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (62) and with the same discussion, there is a contradiction

$$(66) \quad h_2 h_3^{-1} (h_2^{-1} h_3)^3 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have:

$$h_3 h_2 h_3^{-1} h_2^{-3} = 1 \Rightarrow h_3^{-1} h_2^3 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1) \text{ is solvable, a contradiction.}$$

$$(69) \quad h_2 h_3^{-3} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction

$$(73) \quad h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have:

$$h_2 h_3^{-1} h_2^{-1} h_3^{-2} = 1 \Rightarrow h_2^{-1} h_3^{-2} h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1) \text{ is solvable, a contradiction.}$$

$$(76) \quad h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (60) and with the same discussion, there is a contradiction

$$(77) \quad (h_2 h_3^{-2})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (73) and with the same discussion, there is a contradiction

$$(79) \quad h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have:

$$h_3 h_2 h_3^{-1} h_2^3 = 1 \Rightarrow h_3^{-1} h_2^{-3} h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1) \text{ is solvable, a contradiction.}$$

$$(81) \quad (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 = 1:$$

$(h_2 h_3^{-1} h_2)^2 = h_2^{-1} h_3$ and $\langle h_2, h_3 \rangle = \langle h_2 h_3^{-1} h_2, h_2^{-1} h_3 \rangle \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 h_3^{-1} h_2 \rangle$ is abelian, a contradiction.

$$(94) \quad (h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$$h_3^{-3} h_2^{-1} h_3 h_2 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3) \text{ is solvable, a contradiction.}$$

(95) $(h_2 h_3^{-1})^3 h_3^{-1} h_2 = 1$:

Using Tietze transformation where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have:

$$h_2^3 h_3^{-1} h_2 h_3 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^{-3} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3) \text{ is solvable, a contradiction.}$$

(97) $(h_2 h_3^{-1})^4 h_2 = 1$:

$$h_2 = (h_3 h_2^{-1})^4 \text{ and } \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-1} \rangle \text{ is abelian, a contradiction.}$$

(98) $h_3^5 = 1$:

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

(99) $h_3^4 h_2^{-1} h_3 = 1$:

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

(101) $(h_3^2 h_2^{-1})^2 h_3 = 1$:

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

(103) $(h_3 h_2^{-1} h_3)^2 h_2^{-1} h_3 = 1$:

By interchanging h_2 and h_3 in (81) and with the same discussion, there is a contradiction.

(104) $(h_3 h_2^{-1})^4 h_3 = 1$:

By interchanging h_2 and h_3 in (97) and with the same discussion, there is a contradiction.

(105) $(h_2^{-1} h_3)^5 = 1$:

$$(h_2^{-1} h_3)^5 = 1 \text{ and } G \text{ is torsion-free} \Rightarrow h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

C₆ cycles: With the same discussion such as about C_4 cycles and by considering the relations corresponding to some C_6 cycles equivalent, there are 351 non-equivalent cases for the relations of a C_6 cycle in the graph $K(\alpha, \beta)$. These relations are listed in table 5. In the following, we show that some of these relations lead to a contradiction. Each of such relations is marked by a * in the Table 5.

(1) $h_2^6 = 1$:

$$h_2^6 = 1 \text{ and } G \text{ is torsion-free} \Rightarrow h_2 = 1, \text{ a contradiction.}$$

(2) $h_2^5 h_3 = 1$:

$$h_3 = h_2^{-5} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

(3) $h_2^5 h_3^{-1} h_2 = 1$:

$$h_3 = h_2^6 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

(6) $h_2^4 h_3^{-1} h_2^{-1} h_3 = 1$:

$$h_3^{-1} h_2 h_3 = h_2^4 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 4) \text{ is solvable, a contradiction.}$$

(8) $h_2^4 h_3^{-1} h_2 h_3 = 1$:

$$h_3^{-1} h_2 h_3 = h_2^{-4} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -4) \text{ is solvable, a contradiction.}$$

(10) $h_2^2 (h_2 h_3)^2 = 1$:

$$h_2^2 = (h_3^{-1} h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3^{-1} h_2^{-1} \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

(11) $h_2^3 h_3 h_2 h_3^{-1} h_2 = 1$:

$$h_3^{-1} h_2^{-4} h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1) \text{ is solvable, a contradiction.}$$

(15) $h_2^3 h_3 h_2^{-1} h_3^{-1} h_2 = 1$:

$$h_3^{-1} h_2^4 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(4, 1) \text{ is solvable, a contradiction.}$$

(27) $h_2 (h_2^2 h_3^{-1})^2 h_2 = 1$:

$$h_2^2 = (h_3 h_2^{-2})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-2} \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

Table 5: All non-equivalent cases for the relations of a C_6 cycle

n	R	n	R	n	R
1	$h_2^6 = 1 *$	40	$h_2^2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	79	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^3 = 1$
2	$h_2^5 h_3 = 1 *$	41	$h_2^2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	80	$h_2^2 h_3^{-2} h_2^{-2} h_3 = 1$
3	$h_2^5 h_3^{-1} h_2 = 1 *$	42	$h_2^2 h_3^2 h_2 h_3 = 1$	81	$h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1$
4	$h_2^4 h_3^2 = 1$	43	$h_2^2 h_3^2 h_2 h_3^{-1} h_2 = 1$	82	$h_2^2 h_3^{-2} h_2^{-1} h_3^2 = 1$
5	$h_2^4 h_3 h_2^{-1} h_3 = 1$	44	$h_2^2 h_3^4 = 1$	83	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
6	$h_2^4 h_3^{-1} h_2^{-1} h_3 = 1 *$	45	$h_2^2 h_3^3 h_2^{-1} h_3 = 1$	84	$h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$
7	$h_2^4 h_3^{-2} h_2 = 1$	46	$h_2^2 h_3^2 h_2^{-2} h_3 = 1$	85	$h_2^2 h_3^{-4} h_2 = 1$
8	$h_2^4 h_3^{-1} h_2 h_3 = 1 *$	47	$h_2^2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	86	$h_2^2 h_3^{-3} h_2 h_3 = 1$
9	$h_2^3 (h_2 h_3^{-1})^2 h_2 = 1$	48	$h_2^2 h_3^2 h_2^{-1} h_3^2 = 1$	87	$h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
10	$h_2^2 (h_2 h_3)^2 = 1 *$	49	$h_2^2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	88	$h_2^2 h_3^{-2} h_2^2 h_3 = 1$
11	$h_2^3 h_3 h_2 h_3^{-1} h_2 = 1 *$	50	$h_2^2 h_3 h_2^{-3} h_3 = 1$	89	$h_2^2 h_3^{-2} h_2^2 h_3^{-1} h_2 = 1$
12	$h_2^3 h_3^3 = 1$	51	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$	90	$h_2^2 h_3^{-2} h_2 h_3^2 = 1$
13	$h_2^3 h_3^2 h_2^{-1} h_3 = 1$	52	$h_2^2 h_3 h_2^{-2} h_3^2 = 1$	91	$h_2^2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$
14	$h_2^3 h_3 h_2^{-2} h_3 = 1$	53	$h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	92	$h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
15	$h_2^3 h_3 h_2^{-1} h_3^{-1} h_2 = 1 *$	54	$h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	93	$h_2 (h_2 h_3^{-2})^2 h_2 = 1$
16	$h_2^3 h_3 h_2^{-1} h_3^2 = 1$	55	$h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	94	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
17	$h_2^3 (h_3 h_2^{-1})^2 h_3 = 1$	56	$h_2^2 h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1 *$	95	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$
18	$h_2^3 h_3^{-1} h_2^{-2} h_3 = 1$	57	$h_2^2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	96	$(h_2^2 h_3^{-1} h_2)^2 = 1 *$
19	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	58	$h_2^2 h_3 h_2^{-1} h_3 h_2 h_3 = 1$	97	$h_2^2 h_3^{-1} h_2^2 h_3^2 = 1$
20	$h_2^3 h_3^{-1} h_2^{-1} h_3^2 = 1$	59	$h_2^2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	98	$h_2^2 h_3^{-1} h_2^2 h_3 h_2^{-1} h_3 = 1$
21	$h_2^3 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	60	$h_2^2 h_3 h_2^{-1} h_3^3 = 1$	99	$(h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
22	$h_2^3 h_3^{-2} h_2^{-1} h_3 = 1$	61	$h_2^2 (h_3 h_2^{-1} h_3)^2 = 1$	100	$(h_2^2 h_3^{-1})^2 h_3^{-1} h_2 = 1$
23	$h_2^3 h_3^{-3} h_2 = 1$	62	$h_2^2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	101	$(h_2^2 h_3^{-1})^2 h_2 h_3 = 1$
24	$h_2^3 h_3^{-2} h_2 h_3 = 1$	63	$h_2^2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	102	$(h_2^2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1 *$
25	$h_2^3 h_3^{-1} (h_3^{-1} h_2)^2 = 1$	64	$h_2^2 (h_3 h_2^{-1})^2 h_3^2 = 1$	103	$h_2^2 h_3^{-1} (h_2 h_3)^2 = 1$
26	$h_2^3 h_3^{-1} h_2^2 h_3 = 1$	65	$h_2^2 (h_3 h_2^{-1})^3 h_3 = 1$	104	$h_2^2 h_3^{-1} h_2 h_3 h_2 h_3^{-1} h_2 = 1$
27	$h_2 (h_2^2 h_3^{-1})^2 h_2 = 1 *$	66	$h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1$	105	$h_2^2 h_3^{-1} h_2 h_3^3 = 1$
28	$h_2^3 h_3^{-1} h_2 h_3^2 = 1$	67	$h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1$	106	$h_2^2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$
29	$h_2^3 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	68	$h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 = 1$	107	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-2} h_3 = 1$
30	$h_2^2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	69	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1$	108	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^{-1} h_2 = 1 *$
31	$h_2^2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	70	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$	109	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
32	$h_2^2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	71	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$	110	$h_2^2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
33	$h_2^2 (h_2 h_3^{-1})^3 h_2 = 1$	72	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3 = 1$	111	$h_2 (h_2 h_3^{-1})^2 h_2^{-2} h_3 = 1$
34	$(h_2^2 h_3)^2 = 1 *$	73	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1 *$	112	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1$
35	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$	74	$h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1$	113	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
36	$h_2 (h_2 h_3)^2 h_3 = 1$	75	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$	114	$h_2 (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
37	$h_2 (h_2 h_3)^2 h_2^{-1} h_3 = 1$	76	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$	115	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
38	$h_2^2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1 *$	77	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	116	$h_2 (h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1$
39	$h_2^2 h_3 h_2 h_3^{-2} h_2 = 1$	78	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	117	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$

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Table 5 – continued from previous page

n	R	n	R	n	R
118	$h_2^2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$	157	$h_2h_3^5 = 1 *$	196	$(h_2h_3h_2^{-1}h_3)^2 = 1 *$
119	$h_2(h_2h_3^{-1})^2h_2^3h_3 = 1$	158	$h_2h_3^4h_2^{-1}h_3 = 1 *$	197	$h_2h_3h_2^{-1}h_3h_2h_3^{-1}h_2^{-1}h_3 = 1$
120	$h_2^2h_3^{-1}(h_2h_3^{-1}h_2)^2 = 1$	159	$h_2h_3^3h_2^{-2}h_3 = 1$	198	$h_2h_3h_2^{-1}h_3h_2h_3^{-2}h_2 = 1$
121	$h_2(h_2h_3^{-1})^2h_2h_3^2 = 1$	160	$h_2h_3^3h_2^{-1}h_3^{-1}h_2 = 1$	199	$h_2h_3h_2^{-1}h_3h_2h_3^{-1}h_2h_3 = 1$
122	$h_2(h_2h_3^{-1})^2h_2h_3h_2^{-1}h_3 = 1$	161	$h_2h_3^3h_2^{-1}h_3^2 = 1$	200	$h_2h_3h_2^{-1}h_3(h_2h_3^{-1})^2h_2 = 1$
123	$h_2(h_2h_3^{-1})^3h_2^{-1}h_3 = 1$	162	$h_2h_3^2(h_3h_2^{-1})^2h_3 = 1$	201	$h_2h_3h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1$
124	$h_2(h_2h_3^{-1})^3h_3^{-1}h_2 = 1$	163	$h_2h_3^2h_2^{-2}h_3^2 = 1$	202	$h_2h_3h_2^{-1}h_3^4 = 1 *$
125	$h_2(h_2h_3^{-1})^3h_2h_3 = 1$	164	$h_2h_3^2h_2^{-1}(h_2^{-1}h_3)^2 = 1$	203	$h_2h_3h_2^{-1}h_3^3h_2^{-1}h_3 = 1$
126	$h_2(h_2h_3^{-1})^4h_2 = 1$	165	$h_2h_3^2h_2^{-1}h_3^{-1}h_2^{-1}h_3 = 1$	204	$h_2h_3h_2^{-1}h_3^2h_2^{-2}h_3 = 1$
127	$(h_2h_3)^3 = 1 *$	166	$h_2h_3^2h_2^{-1}h_3^{-2}h_2 = 1$	205	$h_2h_3h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$
128	$(h_2h_3)^2h_2h_3^{-1}h_2 = 1$	167	$h_2h_3^2h_2^{-1}h_3^{-1}h_2h_3 = 1$	206	$h_2(h_3h_2^{-1}h_3)^2h_3 = 1$
129	$(h_2h_3)^2h_2^3 = 1 *$	168	$h_2h_3^2h_2^{-1}(h_3^{-1}h_2)^2 = 1$	207	$h_2(h_3h_2^{-1}h_3)^2h_2^{-1}h_3 = 1$
130	$(h_2h_3)^2h_3h_2^{-1}h_3 = 1$	169	$h_2h_3^2h_2^{-1}h_3h_2h_3^{-1}h_2 = 1$	208	$h_2(h_3h_2^{-1})^2h_2^{-1}h_3^2 = 1$
131	$(h_2h_3)^2h_2^{-2}h_3 = 1$	170	$h_2h_3^2h_2^{-1}h_3^3 = 1$	209	$h_2h_3(h_2^{-1}h_3h_2^{-1})^2h_3 = 1$
132	$(h_2h_3)^2h_2^{-1}h_3^{-1}h_2 = 1$	171	$h_2(h_3^2h_2^{-1})^2h_3 = 1$	210	$h_2(h_3h_2^{-1})^2h_3^{-1}h_2^{-1}h_3 = 1$
133	$(h_2h_3)^2h_2^{-1}h_3^2 = 1$	172	$h_2h_3(h_3h_2^{-1})^2h_2^{-1}h_3 = 1$	211	$h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1$
134	$(h_2h_3)^2(h_2^{-1}h_3)^2 = 1$	173	$h_2h_3(h_3h_2^{-1})^2h_3^{-1}h_2 = 1$	212	$h_2(h_3h_2^{-1})^2h_3^{-1}h_2h_3 = 1$
135	$h_2h_3h_2h_3^{-1}h_2^{-2}h_3 = 1$	174	$h_2h_3(h_3h_2^{-1})^2h_3^2 = 1$	213	$h_2(h_3h_2^{-1})^2(h_3^{-1}h_2)^2 = 1$
136	$h_2h_3h_2h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1$	175	$h_2h_3(h_3h_2^{-1})^3h_3 = 1$	214	$h_2(h_3h_2^{-1})^2h_3h_2h_3^{-1}h_2 = 1$
137	$h_2h_3h_2h_3^{-1}h_2^{-1}h_3^2 = 1$	176	$h_2h_3h_2^{-2}h_3h_2h_3^{-1}h_2 = 1$	215	$h_2(h_3h_2^{-1})^2h_3^3 = 1$
138	$h_2h_3h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1$	177	$h_2h_3h_2^{-2}h_3^3 = 1$	216	$h_2h_3h_2^{-1}(h_3h_2^{-1}h_3)^2 = 1$
139	$h_2h_3h_2h_3^{-2}h_2^{-1}h_3 = 1$	178	$h_2h_3h_2^{-2}h_3^2h_2^{-1}h_3 = 1$	217	$h_2(h_3h_2^{-1})^3h_2^{-1}h_3 = 1$
140	$h_2h_3h_2h_3^{-3}h_2 = 1$	179	$h_2(h_3h_2^{-2})^2h_3 = 1$	218	$h_2(h_3h_2^{-1})^3h_3^{-1}h_2 = 1$
141	$h_2h_3h_2h_3^{-2}h_2h_3 = 1$	180	$h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$	219	$h_2(h_3h_2^{-1})^3h_3^2 = 1$
142	$h_2h_3h_2h_3^{-1}(h_3^{-1}h_2)^2 = 1$	181	$h_2h_3h_2^{-1}(h_2^{-1}h_3)^2h_3 = 1$	220	$h_2(h_3h_2^{-1})^4h_3 = 1$
143	$h_2h_3(h_2h_3^{-1}h_2)^2 = 1$	182	$h_2h_3h_2^{-1}(h_2^{-1}h_3)^3 = 1$	221	$(h_2h_3^{-1}h_2^{-1}h_3)^2 = 1 *$
144	$h_2h_3h_2h_3^{-1}h_2h_3^2 = 1$	183	$h_2h_3h_2^{-1}h_3^{-1}h_2^{-1}h_3^2 = 1$	222	$h_2h_3^{-1}h_2^{-1}h_3h_2h_3^{-2}h_2 = 1$
145	$h_2h_3h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	184	$h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$	223	$h_2h_3^{-1}h_2^{-1}h_3h_2h_3^{-1}h_2h_3 = 1$
146	$h_2h_3(h_2h_3^{-1})^2h_2^{-1}h_3 = 1$	185	$h_2h_3h_2^{-1}h_3^{-2}h_2^{-1}h_3 = 1$	224	$h_2h_3^{-1}h_2^{-1}h_3(h_2h_3^{-1})^2h_2 = 1 *$
147	$h_2h_3(h_2h_3^{-1})^2h_3^{-1}h_2 = 1$	186	$h_2h_3h_2^{-1}h_3^{-3}h_2 = 1$	225	$h_2h_3^{-1}h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1$
148	$h_2h_3(h_2h_3^{-1})^2h_2h_3 = 1$	187	$h_2h_3h_2^{-1}h_3^{-2}h_2h_3 = 1$	226	$h_2h_3^{-1}h_2^{-1}h_3^4 = 1 *$
149	$h_2h_3(h_2h_3^{-1})^3h_2 = 1$	188	$h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$	227	$h_2h_3^{-1}h_2^{-1}h_3^3h_2^{-1}h_3 = 1 *$
150	$(h_2h_3^2)^2 = 1 *$	189	$h_2h_3h_2^{-1}h_3^{-1}h_2^2h_3^{-1}h_2 = 1$	228	$h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-2}h_3 = 1$
151	$h_2h_3^2h_2h_3h_2^{-1}h_3 = 1$	190	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^2 = 1 *$	229	$h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$
152	$h_2h_3^2h_2h_3^{-1}h_2^{-1}h_3 = 1 *$	191	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	230	$h_2h_3^{-1}(h_2^{-1}h_3^2)^2 = 1$
153	$h_2h_3^2h_2h_3^{-2}h_2 = 1$	192	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$	231	$h_2h_3^{-1}h_2^{-1}h_3(h_3h_2^{-1})^2h_3 = 1 *$
154	$h_2h_3^2h_2h_3^{-1}h_2h_3 = 1$	193	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-2}h_2 = 1$	232	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-2}h_3^2 = 1$
155	$h_2h_3^2(h_2h_3^{-1})^2h_2 = 1$	194	$h_2h_3h_2^{-1}(h_3^{-1}h_2)^2h_3 = 1$	233	$h_2h_3^{-1}(h_2^{-1}h_3h_2^{-1})^2h_3 = 1$
156	$h_2h_3^3h_2h_3^{-1}h_2 = 1$	195	$h_2h_3h_2^{-1}(h_3^{-1}h_2)^3 = 1$	234	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2^{-1}h_3 = 1$

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Table 5 – continued from previous page

n	R	n	R	n	R
235	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1$	274	$h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	313	$(h_2 h_3^{-1})^2 h_3^{-2} h_2 h_3 = 1$
236	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$	275	$(h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1$	314	$(h_2 h_3^{-1})^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$
237	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	276	$(h_2 h_3^{-2})^2 h_3^{-1} h_2 = 1$	315	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3^2 = 1$
238	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1$	277	$(h_2 h_3^{-2})^2 h_2 h_3 = 1$	316	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
239	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1$	278	$(h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2 = 1$	317	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
240	$h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	279	$h_2 h_3^{-2} (h_2 h_3^{-1} h_2)^2 = 1$	318	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$
241	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-2} h_3 = 1 *$	280	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1$	319	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3 = 1$
242	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$	281	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$	320	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1$
243	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1$	282	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1 *$	321	$((h_2 h_3^{-1})^2 h_2)^2 = 1 *$
244	$h_2 h_3^{-1} (h_2^{-1} h_3)^4 = 1 *$	283	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1$	322	$(h_2 h_3^{-1})^2 h_2 h_3^3 = 1$
245	$h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	284	$h_2 h_3^{-1} (h_3^{-1} h_2)^3 h_3 = 1$	323	$(h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1$
246	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$	285	$h_2 h_3^{-1} (h_3^{-1} h_2)^4 = 1 *$	324	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3^2 = 1$
247	$h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$	286	$(h_2 h_3^{-1} h_2)^3 = 1 *$	325	$(h_2 h_3^{-1})^2 h_2 (h_3 h_2^{-1})^2 h_3 = 1$
248	$h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1$	287	$(h_2 h_3^{-1} h_2)^2 h_3^2 = 1$	326	$(h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1$
249	$h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	288	$(h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$	327	$(h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$
250	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 h_3 = 1$	289	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1$	328	$(h_2 h_3^{-1})^3 h_3^{-1} h_2^{-1} h_3 = 1$
251	$h_2 h_3^{-2} (h_2^{-1} h_3)^3 = 1$	290	$(h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1$	329	$(h_2 h_3^{-1})^3 h_3^{-2} h_2 = 1$
252	$h_2 h_3^{-3} h_2^{-1} h_3^2 = 1$	291	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 h_3 = 1$	330	$(h_2 h_3^{-1})^3 h_3^{-1} h_2 h_3 = 1$
253	$h_2 h_3^{-3} (h_2^{-1} h_3)^2 = 1$	292	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 h_3^{-1} h_2 = 1 *$	331	$h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
254	$h_2 h_3^{-4} h_2^{-1} h_3 = 1 *$	293	$(h_2 h_3^{-1} h_2 h_3)^2 = 1 *$	332	$(h_2 h_3^{-1})^3 h_2 h_3^2 = 1$
255	$h_2 h_3^{-5} h_2 = 1$	294	$h_2 h_3^{-1} h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	333	$(h_2 h_3^{-1})^3 h_2 h_3 h_2^{-1} h_3 = 1$
256	$h_2 h_3^{-4} h_2 h_3 = 1$	295	$h_2 h_3^{-1} h_2 h_3^4 = 1$	334	$(h_2 h_3^{-1})^4 h_2^{-1} h_3 = 1 *$
257	$h_2 h_3^{-3} (h_3^{-1} h_2)^2 = 1$	296	$h_2 h_3^{-1} h_2 h_3^3 h_2^{-1} h_3 = 1$	335	$(h_2 h_3^{-1})^4 h_3^{-1} h_2 = 1 *$
258	$h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$	297	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3^2 = 1$	336	$(h_2 h_3^{-1})^4 h_2 h_3 = 1$
259	$h_2 h_3^{-3} h_2 h_3^2 = 1$	298	$h_2 h_3^{-1} h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	337	$(h_2 h_3^{-1})^5 h_2 = 1 *$
260	$h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1$	299	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^3 = 1$	338	$h_3^6 = 1 *$
261	$h_2 h_3^{-3} h_2 h_3^{-1} h_2^{-1} h_3 = 1 *$	300	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1} h_3)^2 = 1$	339	$h_3^5 h_2^{-1} h_3 = 1 *$
262	$h_2 h_3^{-1} (h_3^{-2} h_2)^2 = 1$	301	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	340	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$
263	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1$	302	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	341	$h_3 (h_3^2 h_2^{-1})^2 h_3 = 1 *$
264	$h_2 h_3^{-2} (h_3^{-1} h_2)^3 = 1$	303	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	342	$h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$
265	$(h_2 h_3^{-2} h_2)^2 = 1 *$	304	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^3 h_3 = 1$	343	$(h_3^2 h_2^{-1} h_3)^2 = 1 *$
266	$h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1$	305	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^3 = 1$	344	$(h_3^2 h_2^{-1})^2 h_3 h_2^{-1} h_3 = 1 *$
267	$h_2 h_3^{-2} h_2 (h_2 h_3^{-1})^2 h_2 = 1$	306	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$	345	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
268	$h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1$	307	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_3 = 1$	346	$h_3 (h_3 h_2^{-1})^4 h_3 = 1$
269	$h_2 h_3^{-2} h_2 h_3^3 = 1$	308	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$	347	$(h_3 h_2^{-1} h_3)^3 = 1 *$
270	$h_2 h_3^{-2} h_2 h_3^2 h_2^{-1} h_3 = 1$	309	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	348	$(h_3 h_2^{-1} h_3)^2 h_2^{-1} h_3 h_2^{-1} h_3 = 1 *$
271	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1$	310	$(h_2 h_3^{-1})^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	349	$((h_3 h_2^{-1})^2 h_3)^2 = 1 *$
272	$h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	311	$(h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1$	350	$(h_3 h_2^{-1})^5 h_3 = 1 *$
273	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$	312	$(h_2 h_3^{-1})^2 h_3^{-3} h_2 = 1$	351	$(h_2^{-1} h_3)^6 = 1 *$

$$(34) (h_2^2 h_3)^2 = 1:$$

$(h_2^2 h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_2^2 h_3 = 1 \Rightarrow h_3 = h_2^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(38) h_2^2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2^{-1} h_3$ and $h_2 \mapsto h_2$, we have $h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1$.

Using Tietze transformation again where $h_3 \mapsto h_2^{-1} h_3$ and $h_2 \mapsto h_2$, we have:

$h_3 h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3^2 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(56) h_2^2 h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2^{-1} h_3$ and $h_2 \mapsto h_2$, we have $h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$.

Using Tietze transformation again where $h_3 \mapsto h_2^{-1} h_3$ and $h_2 \mapsto h_2$, we have:

$h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(73) h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1$.

Using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1$. Now by using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(96) (h_2^2 h_3^{-1} h_2)^2 = 1:$$

$(h_2^2 h_3^{-1} h_2)^2 = 1$ and G is torsion-free $\Rightarrow h_2^2 h_3^{-1} h_2 = 1 \Rightarrow h_3 = h_2^3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(102) (h_2^2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $(h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$. Using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$(h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.

$$(108) h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^{-1} = 1$.

Using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^{-1} = 1$. Now by using Tietze transformation again where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$h_3^{-1} h_2 h_3 h_2^{-1} h_3^{-1} = 1 \Rightarrow h_2^{-1} h_3^2 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(127) (h_2 h_3)^3 = 1:$$

$(h_2 h_3)^3 = 1$ and G is torsion-free $\Rightarrow h_2 h_3 = 1 \Rightarrow h_3 = h_2^{-1} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(129) (h_2 h_3)^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(150) (h_2 h_3^2)^2 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(152) h_2 h_3^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(157) h_2 h_3^5 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(158) \quad h_2 h_3^4 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(190) \quad h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.

$$(196) \quad (h_2 h_3 h_2^{-1} h_3)^2 = 1:$$

$(h_2 h_3 h_2^{-1} h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^{-1} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(202) \quad h_2 h_3 h_2^{-1} h_3^4 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(221) \quad (h_2 h_3^{-1} h_2^{-1} h_3)^2 = 1:$$

$(h_2 h_3^{-1} h_2^{-1} h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(224) \quad h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_2 h_3^{-1} h_2^{-1} h_3 h_2 (h_3^{-1})^2 = 1$.

Using Tietze transformation again where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have $h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 h_2 = 1$. Now by using Tietze transformation again where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have:

$h_2 h_3^{-1} h_2^{-1} h_3 h_2 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(226) \quad h_2 h_3^{-1} h_2^{-1} h_3^4 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(227) \quad h_2 h_3^{-1} h_2^{-1} h_3^3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (108) and with the same discussion, there is a contradiction.

$$(231) \quad h_2 h_3^{-1} h_2^{-1} h_3 (h_3 h_2^{-1})^2 h_3 = 1:$$

By using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $h_3^{-1} h_2^{-1} h_3 h_2 h_3^3 h_2 = 1$.

Using Tietze transformation again where $h_2 \mapsto h_3^{-1} h_2$ and $h_3 \mapsto h_3$, we have $h_3^{-1} h_2^{-1} h_3 h_2 h_3^2 h_2 = 1$. Now by using Tietze transformation again where $h_2 \mapsto h_3^{-1} h_2$ and $h_3 \mapsto h_3$, we have $h_3^{-1} h_2^{-1} h_3 h_2 h_3 h_2 = 1$. By using Tietze transformation again where $h_2 \mapsto h_3^{-1} h_2$ and $h_3 \mapsto h_3$, we have $h_3^{-1} h_2^{-1} h_3 h_2^2 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(241) \quad h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (231) and with the same discussion, there is a contradiction.

$$(244) \quad h_2 h_3^{-1} (h_2^{-1} h_3)^4 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have:

$h_2 h_3^{-1} h_2^{-1} h_3^4 = 1 \Rightarrow h_2^{-1} h_3^4 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(4, 1)$ is solvable, a contradiction.

$$(254) \quad h_2 h_3^{-4} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(261) \quad h_2 h_3^{-3} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (73) and with the same discussion, there is a contradiction.

$$(265) \quad (h_2 h_3^{-2} h_2)^2 = 1:$$

$(h_2 h_3^{-2} h_2)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-2} h_2 = 1 \Rightarrow h_2^2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(282) \quad h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (224) and with the same discussion, there is a contradiction.

$$(285) \quad h_2 h_3^{-1} (h_3^{-1} h_2)^4 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have:

$$h_3 h_2 h_3^{-1} h_2^4 = 1 \Rightarrow h_3^{-1} h_2^{-4} h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1) \text{ is solvable, a contradiction.}$$

$$(286) \quad (h_2 h_3^{-1} h_2)^3 = 1:$$

$(h_2 h_3^{-1} h_2)^3 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-1} h_2 = 1 \Rightarrow h_3 = h_2^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(292) \quad (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have $(h_2 h_3^{-1})^2 h_3^{-2} = 1$. Using Tietze transformation again where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have $h_2^2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(293) \quad (h_2 h_3^{-1} h_2 h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (196) and with the same discussion, there is a contradiction.

$$(321) \quad ((h_2 h_3^{-1})^2 h_2)^2 = 1:$$

$((h_2 h_3^{-1})^2 h_2)^2 = 1$ and G is torsion-free $\Rightarrow (h_2 h_3^{-1})^2 h_2 = 1 \Rightarrow h_2 = (h_3 h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.

$$(334) \quad (h_2 h_3^{-1})^4 h_2^{-1} h_3 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have:

$$h_2^4 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^4 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 4) \text{ is solvable, a contradiction.}$$

$$(335) \quad (h_2 h_3^{-1})^4 h_3^{-1} h_2 = 1:$$

Using Tietze transformation where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have:

$$h_2^4 h_3^{-1} h_2 h_3 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2^{-4} \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -4) \text{ is solvable, a contradiction.}$$

$$(337) \quad (h_2 h_3^{-1})^5 h_2 = 1:$$

$h_2 = (h_3 h_2^{-1})^5 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.

$$(338) \quad h_3^6 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(339) \quad h_3^5 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(341) \quad h_3 (h_3^2 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(343) \quad (h_3^2 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (96) and with the same discussion, there is a contradiction.

$$(344) \quad (h_3^2 h_2^{-1})^2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (102) and with the same discussion, there is a contradiction.

$$(347) \quad (h_3 h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (286) and with the same discussion, there is a contradiction.

$$(348) \quad (h_3 h_2^{-1} h_3)^2 h_2^{-1} h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (292) and with the same discussion, there is a contradiction.

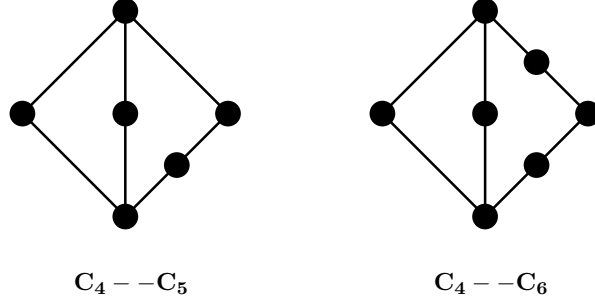


FIGURE 14. $(C_4 - -C_5)$ A C_4 and a C_5 cycle with two common edges, and $(C_4 - -C_6)$ A C_4 and a C_6 cycle with two common edges in $K(\alpha, \beta)$

$$(349) \quad ((h_3 h_2^{-1})^2 h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (321) and with the same discussion, there is a contradiction.

$$(350) \quad (h_3 h_2^{-1})^5 h_3 = 1:$$

By interchanging h_2 and h_3 in (337) and with the same discussion, there is a contradiction.

$$(351) \quad (h_2^{-1} h_3)^6 = 1:$$

$(h_2^{-1} h_3)^6 = 1$ and G is torsion-free $\Rightarrow h_2^{-1} h_3 = 1 \Rightarrow h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

8.1. **$K_{2,3}$.** By Theorem 2.17, the Kaplansky graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the complete bipartite graph $K_{2,3}$.

8.2. **$C_4 - -C_5$.** Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4]$ is the 8-tuple related to the cycle C_4 and $[h'_1, h''_1, h'_2, h''_2, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7]$ is the 10-tuple related to the cycle C_5 in the graph $C_4 - -C_5$, where the first four components of these tuples are related to the common edges of C_4 and C_5 . Without loss of generality we may assume that $h'_1 = 1$ and $\alpha = 1 + h_2 + h_3$. It can be seen that there are 121 different cases for the relations of the cycles C_4 and C_5 in this structure. Using GAP [9], we see that the groups with two generators h_2 and h_3 and two relations which are between 111 cases of these 121 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 10 cases for the relations of the cycles C_4 and C_5 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - -C_5$ in $K(\alpha, \beta)$. Such cases are listed in table 6. In the following, we show that all of these 10 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - -C_5$.

$$(1) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$$R_1 : h_2^3 = h_3^{-2} \text{ and } R_2 : h_2^3 = h_3^2 \Rightarrow h_3^4 = 1 \text{ and } G \text{ is torsion-free} \Rightarrow h_3 = 1, \text{ a contradiction.}$$

$$(2) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$$R_1 : h_2^3 = h_3^{-2} \text{ and } R_2 : h_2 h_3^{-2} h_2^{-1} h_3^{-2} = 1 \Rightarrow h_2 h_2^3 h_2^{-1} h_2^3 = 1 \Rightarrow h_2^6 = 1 \text{ and } G \text{ is torsion-free} \Rightarrow h_2 = 1, \text{ a contradiction.}$$

TABLE 6. The relations of a $C_4 - -C_5$ in the Kaplansky graph

n	R_1	R_2
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
3	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^3 h_3^2 = 1$
4	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
5	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$
6	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^3 = 1$
7	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
8	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
9	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
10	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$

(3) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^3 h_3^2 = 1$:

$R_1 : h_2^3 = h_3^2$ and $R_2 : h_2^3 = h_3^{-2} \Rightarrow h_3^4 = 1$ and G is torsion-free $\Rightarrow h_3 = 1$, a contradiction.

(4) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:

Using Tietze transformation where $h_2 \mapsto h_3 h_2$ and $h_3 \mapsto h_3$, we have $R_1 : h_3 h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. Using Tietze transformation again where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have $R_1 : h_3 h_2 h_3^{-1} = h_2^{-1} h_3$ and $R_2 : (h_2^{-1} h_3)^2 (h_2 h_3^{-1})^2 = 1$. Using R_1 and R_2 , we have $(h_3 h_2 h_3^{-1})^2 = (h_3 h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2 h_3^{-1}, h_3 h_2^{-1} \rangle \cong BS(1, -1)$ is solvable, a contradiction.

(5) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have:

$R_1 : h_2^2 = h_3^{-3}$ and $R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1 \Rightarrow h_3^6 = 1$ and G is torsion-free $\Rightarrow h_3 = 1$, a contradiction.

(6) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^3 = 1$:

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

(7) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

(8) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

(9) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$:

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

(10) $R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

8.3. **$C_4 - -C_6$.** Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4]$ is the 8-tuple related to the cycle C_4 and $[h'_1, h''_1, h'_2, h''_2, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8]$ is the 12-tuple related to the cycle C_6 in the graph $C_4 - -C_6$, where the first four components of these tuples are related to the common edges of C_4 and C_6 . Without loss of generality we may assume that $h'_1 = 1$ and $\alpha = 1 + h_2 + h_3$. Also it is easy to see that $h'_3 \neq h'_5$ and $h''_4 \neq h''_8$. With these assumptions and by considering the relations from Tables 2 and 5 which are

not disproved, it can be seen that there are 658 different cases for the relations of the cycles C_4 and C_6 in this structure. By considering all groups with two generators h_2 and h_3 and two relations which are between these cases and by using GAP [9], we see that 632 groups are finite and solvable, or just finite. So, there are just 20 cases for the relations of the cycles C_4 and C_6 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - -C_6$ in $K(\alpha, \beta)$. These cases are listed in table 7. In the following, we show that all of such 20 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - -C_6$.

TABLE 7. The relations of a $C_4 - -C_6$ in the Kaplansky graph

n	R_1	R_2
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
3	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
4	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$
5	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1$
6	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
7	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
8	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
9	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1$
10	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1$
11	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
12	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
13	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
14	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1$
15	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$
16	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
17	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3 = 1$
18	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$
19	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$(h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
20	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^2 h_3^{-1} h_2 = 1$

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
 $R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1 \Rightarrow h_2 h_3 h_2^{-1} = h_3^2 h_2 h_3 (*)$. Using R_1 and $(*)$ we have $(h_3^2 h_2)^2 = 1$ and G is torsion-free $\Rightarrow h_3^2 h_2 = 1 \Rightarrow h_2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (2) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
 $R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 h_2 h_3^2 (*)$. Using R_1 and $(*)$ we have $(h_2 h_3^2)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^2 = 1 \Rightarrow h_2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $R_1 : h_2^2 h_3^{-2} h_2 = 1 \Rightarrow h_2^2 h_3^{-1} = h_2^{-1} h_3 (*)$ and $h_3^2 = h_3^2 (**)$. $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 h_2 = 1 \Rightarrow h_3^2 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1 (***)$. Using $(**)$ and $(***)$ we

- have $h_2^2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1 \Rightarrow$ By $(*)$, $h_2^{-1} h_3 h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} = 1 \Rightarrow (h_2^{-1} h_3 h_2 h_3^{-1})^2 = 1$ and G is torsion-free $\Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (4) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$:
 $R_1 : h_2^2 h_3^{-2} h_2 = 1 \Rightarrow h_2^{-2} = h_2 h_3^{-2} (*)$ and $h_3^{-2} = h_2^{-3} (**)$. Using R_2 and $(*)$ we have $(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$. By $(**)$, $(h_2 h_3^{-1})(h_3^{-1} h_2) = h_2 h_3^{-2} h_2 = h_2^{-1} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 h_3^{-1}, h_3^{-1} h_2 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (5) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1$:
 $R_1 : h_2^2 h_3^{-2} h_2 = 1 \Rightarrow h_2^{-2} = h_2 h_3^{-2} (*)$. Using R_2 and $(*)$ we have $h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_3 h_2^{-1} h_3 = 1 \Rightarrow (h_2 h_3^{-1} h_2^{-1} h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
 $R_1 : h_2^2 h_3^{-2} h_2 = 1 \Rightarrow h_2^2 = h_3^2 h_2^{-1} (*)$ and $h_3^2 = h_2^3 (**)$. Using R_2 , $(*)$ and $(**)$ we have $(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$. By $(**)$, $(h_2 h_3^{-1})(h_3^{-1} h_2) = h_2 h_3^{-2} h_2 = h_2^{-1} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 h_3^{-1}, h_3^{-1} h_2 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (7) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
 $R_1 : h_2 h_3 h_2^{-2} h_3 = 1 \Rightarrow h_3 h_2 h_3 = h_2^2$ and $R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_3 h_2^{-1} h_3^{-1} h_2 = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (8) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $R_1 : h_2 h_3 h_2^{-2} h_3 = 1 \Rightarrow h_2^{-1} h_3^{-1} h_2^2 h_3^{-1} = 1 (*)$. Using R_2 and $(*)$ we have $h_2^{-1} h_3^2 = 1 \Rightarrow h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1$, $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1$:
 $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1 \Rightarrow h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^2 = 1 \Rightarrow h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3^{-2} = 1 \Rightarrow h_2 h_3 h_2^{-1} = h_3^2 h_2^{-1} h_3^2 (*)$. By using R_1 and $(*)$ we have $(h_3^3 h_2^{-1})^2 = 1$ and G is torsion-free $\Rightarrow h_3^3 h_2^{-1} = 1 \Rightarrow h_2 = h_3^3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (10) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1$, $R_2 : (h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1$:
 $R_2 : (h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3^2 h_2^{-1} h_3^2 (*)$. By using R_1 and $(*)$ we have $(h_2^{-1} h_3^3)^2 = 1$ and G is torsion-free $\Rightarrow h_2^{-1} h_3^3 = 1 \Rightarrow h_2 = h_3^3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (11) $R_1 : h_2 h_3^{-3} h_2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (12) $R_1 : h_2 h_3^{-3} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (13) $R_1 : h_2 h_3^{-3} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.
- (14) $R_1 : h_2 h_3^{-3} h_2 = 1$, $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1$:
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (15) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.
- (16) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(17) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(18) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(19) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : (h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(20) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$C_4 - C_5$ subgraph: Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4]$ is the 8-tuple related to the cycle C_4 and $[h'_1, h''_1, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8]$ is the 10-tuple related to the cycle C_5 in the graph $C_4 - C_5$, where the first two components of these tuples are related to the common edge of C_4 and C_5 . With the same argument such as about $C_4 - C_5$, without loss of generality we may assume that $h'_1 = 1$, where $h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. Also it is easy to see that $h'_2 \neq h'_5$ and $h''_4 \neq h''_8$. With these assumptions and by considering the relations from Tables 2 and 4 which are not disproved, it can be seen that there are 482 cases for the relations of the cycles C_4 and C_5 in this structure. Using Gap [9], we see that all groups with two generators h_2 and h_3 and two relations which are between 426 cases of these 482 cases are finite and solvable, that is a contradiction with the assumptions. So there are just 56 cases for the relations of the cycles C_4 and C_5 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5$ in the graph $K(\alpha, \beta)$. These cases are listed in table 8. In the following, we show that 50 cases of these relations lead to a contradiction and just 6 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 8.

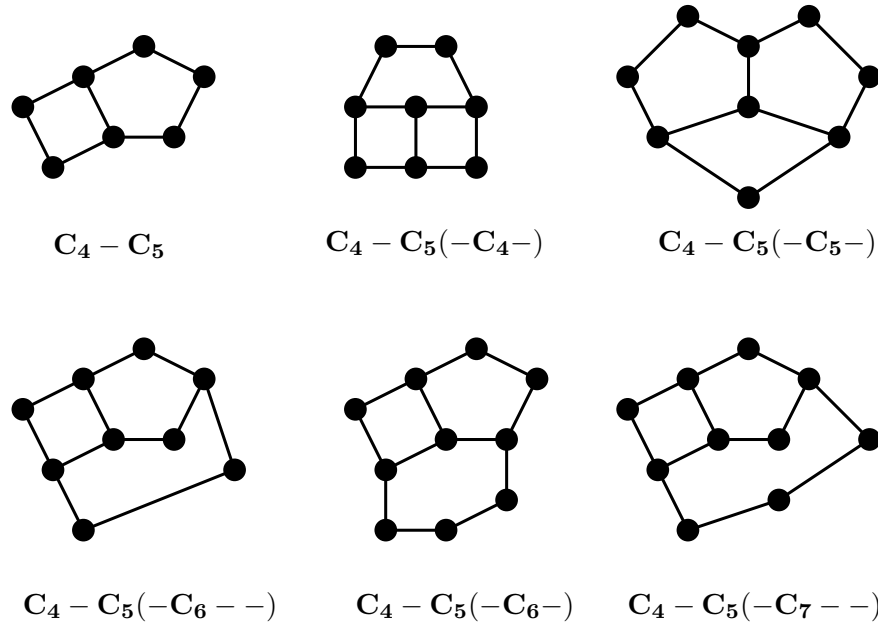


FIGURE 15. The graph $C_4 - C_5$ and some forbidden subgraphs which contain such graph

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$:
 $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_2^2 = h_3^{-1} h_2 h_3^{-1} (*) \Rightarrow R_2 : h_2 h_3^{-1} h_2 h_3^{-1} h_2 h_3^{-1} h_2 h_3^{-1} = 1 \Rightarrow (h_2 h_3^{-1})^4 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-1} = 1 \Rightarrow h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (2) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1 \Rightarrow h_2^2 h_3 = h_3 h_2^2 (*) \Rightarrow R_1 : h_3 h_2^2 h_2^{-1} h_3 = 1 \Rightarrow h_2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_2^2 = h_3^{-1} h_2 h_3^{-1} (*) \Rightarrow R_2 : h_2^6 = 1 \Rightarrow h_2 = 1$, a contradiction.
- (4) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_2^3 h_2^2 = h_2^2 h_3^2 (*)$ and $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow (h_3 h_2^2 h_3 h_2^{-1})(h_2^2 h_3 h_2^{-1} h_3) = 1 (**)$. By $(*)$ and $(**)$ we have $h_2 h_3^3 h_2 h_3 = 1 \Rightarrow h_3^2 (h_2 h_3)^2 = 1 \Rightarrow h_3^2 = (h_3^{-1} h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3, h_3^{-1} h_2^{-1} \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (5) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$:
 $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_2 h_3 h_2^{-1} h_3 = h_2^{-1} (*) \Rightarrow R_2 : h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$:
 $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_3 h_2^{-1} h_3 h_2 = h_2^{-1} (*) \Rightarrow R_2 : h_3^{-1} h_2 h_3 h_2^{-1} = 1 \Rightarrow h_3^{-1} h_2 h_3 = h_2 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (9) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$:
 $R_1 : h_2^2 h_3^{-2} h_2 = 1 \Rightarrow h_3^{-2} = h_2^{-3} (*)$ and $R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2 h_3 = 1 (**)$. By $(*)$ and $(**)$ we have $h_2^{-1} h_3^{-1} h_2 h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (10) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. $R_2 \Rightarrow h_2^2 h_3^{-2} = h_3^{-2} h_2^2 (*)$ and $R_1 \Rightarrow (h_3^{-1} h_2^2 h_3^{-1} h_2^{-1})(h_2^2 h_3^{-1} h_2^{-1} h_3^{-1}) = 1 (**)$. By $(*)$ and $(**)$ we have $h_2 h_3^{-3} h_2 h_3^{-1} = 1 \Rightarrow h_3^{-2} (h_2 h_3^{-1})^2 = 1 \Rightarrow h_3^2 = (h_2 h_3^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3, h_2 h_3^{-1} \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (11) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$:
 $R_1 \Rightarrow h_3^{-2} = h_2^{-3} (*)$ and $R_2 : h_2 h_3^{-1} h_2 h_3^{-2} h_2 h_3 = 1 (**)$. By $(*)$ and $(**)$ we have $h_2 h_3^{-1} h_2 h_3^{-3} h_2 h_3 = 1 \Rightarrow h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (12) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^3 h_3^2 = 1$:
 $R_1 \Rightarrow h_3^3 = h_2^2$ and $R_2 \Rightarrow h_2^3 = h_3^{-2} \Rightarrow h_3^4 = 1 \Rightarrow h_3 = 1$, a contradiction.
- (13) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $R_1 \Rightarrow h_2^2 h_3^{-1} = h_2^{-1} h_3 (*) \Rightarrow R_2 : h_2^{-1} h_3 h_2^{-2} h_3 = 1 \Rightarrow h_2 = (h_2^{-1} h_3)^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_2^{-1} h_3 \rangle = \langle h_2^{-1} h_3 \rangle$ is abelian, a contradiction.
- (14) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 $R_1 \Rightarrow h_3^3 = h_2^2 (*) \Rightarrow R_2 : h_2^6 = 1 \Rightarrow h_2 = 1$, a contradiction.
- (15) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $R_2 \Rightarrow h_2^{-2} h_3 = h_3 h_2^{-2} (*) \Rightarrow R_1 : h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

TABLE 8. The relations of a $C_4 - C_5$ in the Kaplansky graph

n	R_1	R_2	n	R_1	R_2
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	29	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	30	$h_2 h_3^{-3} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
3	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	31	$h_2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
4	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	32	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
5	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$	33	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^3 h_3^{-2} h_2 = 1$
6	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	34	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
7*	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	35	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$
8*	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	36	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
9	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	37	$h_2 h_3^{-2} h_2 h_3 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
10	$h_2^2 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	38	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
11	$h_2^2 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$	39	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
12	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^3 h_3^2 = 1$	40	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
13	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	41*	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
14	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	42	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
15	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	43	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
16	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	44	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
17	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	45	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3 h_2^{-1} h_3 = 1$
18	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	46	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
19	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	47	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
20*	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	48	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$
21	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	49	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
22	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	50	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
23	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	51	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
24	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3 h_2^{-2} h_3^2 = 1$	52	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$
25	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	53	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$
26	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^3 = 1$	54	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
27*	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	55*	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
28	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	56	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$

(16) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$

$R_2 \Rightarrow h_2^{-2} = h_3^{-4} (*) \Rightarrow R_1 : h_2 h_3 h_3^{-4} h_3 = 1 \Rightarrow h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

(17) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$

$R_1 \Rightarrow h_2 h_3^{-1} h_2^{-1} = h_2^{-1} h_3 (*)$ and $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^2 = 1 (**)$. By $(*)$ and $(**)$ we have $(h_2^{-1} h_3^2)^2 = 1$ and G is torsion-free $\Rightarrow h_2^{-1} h_3^2 = 1 \Rightarrow h_2 = h_3^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

- (18) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 $R_2 \Rightarrow h_2 h_3^2 = h_3^2 h_2$ (*) and $R_1 \Rightarrow (h_3 h_2 h_3 h_2^{-2})(h_2 h_3 h_2^{-2} h_3) = 1$ (**). By (*) and (**) we have $h_2^{-1} h_3^3 h_2^{-1} h_3 = 1 \Rightarrow h_3^2 = (h_3^{-1} h_2)^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3, h_3^{-1} h_2 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (19) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1$:
 $R_1 \Rightarrow h_3 h_2 h_3 h_2^{-1} = h_2$ (*) and $R_2 : h_2 h_3 h_2^{-1} h_3 h_2^{-1} h_3 h_2^{-1} h_3 = 1$ (**). By (*) and (**) we have $h_2 = (h_2 h_3^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_2 h_3^{-1} \rangle = \langle h_2 h_3^{-1} \rangle$ is abelian, a contradiction.
- (21) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:
Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^2 h_3^3 = 1$ and $R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$. $R_1 \Rightarrow h_2^2 = h_3^{-3}$ (*) $\Rightarrow R_2 : h_3^6 = 1 \Rightarrow h_3 = 1$, a contradiction.
- (22) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^3 h_2 = 1$:
 $R_1 \Rightarrow (h_3^{-1} h_2)^2 h_3^{-1} = h_2$ (*) and $R_2 : h_2^3 (h_3^{-1} h_2)^2 h_3^{-1} = 1$ (**). By (*) and (**) we have $h_2^4 = 1 \Rightarrow h_2 = 1$, a contradiction.
- (23) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-2} h_3 = 1$:
 $R_1 \Rightarrow h_2^{-1} h_3 h_2 h_3 = h_3^{-1} h_2$ (*) $\Rightarrow R_2 : h_2^{-1} h_3^{-1} h_2 h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (24) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-2} h_3^2 = 1$:
 $R_1 \Rightarrow h_3 h_2 h_3 h_2^{-1} = h_2 h_3^{-1}$ (*) $\Rightarrow R_2 : h_2^{-1} h_3 h_2 h_3^{-1} = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (25) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
Using Tietze transformation where $h_2 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. With the same discussion such as (18), there is a contradiction.
- (26) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^3 = 1$:
 $R_1 \Rightarrow h_2^2 = h_3^3$ and $R_2 \Rightarrow h_2^2 = h_3^{-3}$. $\Rightarrow h_3^6 = 1 \Rightarrow h_3 = 1$, a contradiction.
- (28) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:
 $R_1 \Rightarrow h_2^2 = h_3^3$ (*) $\Rightarrow R_2 : h_3^6 = 1 \Rightarrow h_3 = 1$, a contradiction.
- (29) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.
- (30) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.
- (31) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.
- (32) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.
- (33) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_3^2 h_3^{-2} h_2 = 1$:
By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.
- (34) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.
- (35) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$:
By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(36) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(37) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(38) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(39) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(40) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(42) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(43) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(44) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(45) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^3 h_3 h_2^{-1} h_3 = 1:$$

$R_2 \Rightarrow h_2 h_3 h_2^{-1} h_3 = h_2^{-2} (*) \Rightarrow R_1 : h_3^{-1} h_2^{-1} = 1 \Rightarrow h_3 = h_2^{-1} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(46) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$R_2 \Rightarrow h_2^{-1} h_3 h_2 = h_2 h_3 h_2^{-1} (*)$ and $h_3^{-1} h_2^{-2} = h_2^{-2} h_3^{-1} (**)$. $(*) \Rightarrow R_1 : h_3^{-1} h_2 h_3 h_2 h_3 h_2^{-1} = 1 \Rightarrow (h_2^{-1} h_3^{-1} h_2 h_3 h_2 h_3)(h_3^{-1} h_2 h_3 h_2 h_3 h_2^{-1}) = 1 (***)$. By $(**)$ and $(***)$ we have $h_2^2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3^{-1} \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(47) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$R_1 \Rightarrow h_3 h_2^{-1} h_3^{-1} h_2 = h_2 h_3 (*) \Rightarrow R_2 : (h_2 h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_3 = h_2^{-1} \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(48) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(49) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(50) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(51) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.

$$(52) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(53) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(54) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(56) \ R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \ R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

8.4. **$C_4 - C_5(-C_5-)$.** It can be seen that there are 42 cases for the relations of a cycle C_4 and two cycles C_5 in the graph $C_4 - C_5(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 38 cases of these 42 cases are finite and solvable, that is a contradiction. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_5-)$ in $K(\alpha, \beta)$. Such cases are listed in table 9. In the following it can be seen that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_5-)$.

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_3^2 h_2^2 = h_2^2 h_3^2 (*)$ and $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1 \Rightarrow (h_3 h_2^2 h_3 h_2^{-1})(h_2^2 h_3 h_2^{-1} h_3) = 1 (**)$. By $(*)$ and $(**)$ we have $h_2 h_3^3 h_2 h_3 = 1 \Rightarrow h_3^2 (h_2 h_3)^2 = 1 \Rightarrow h_3^2 = (h_3^{-1} h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3, h_3^{-1} h_2^{-1} \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $R_1 \Rightarrow h_2^3 = h_3^2 (*) \Rightarrow R_3 : h_2^6 = 1 \Rightarrow h_2 = 1$, a contradiction.
- (3) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (4) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

TABLE 9. All the relations related to the existence of a $C_4 - C_5(-C_5-)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
2	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
3	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$
4	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$

8.5. **$C_4 - C_5(-C_4-)$.** It can be seen that there are 4 cases for the relations of two cycles C_4 and a cycle C_5 in this structure. By considering all groups with two generators h_2 and h_3 and three relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_4-)$.

8.6. **$C_4 - C_5(-C_6-)$.** It can be seen that there are 126 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_6 in the graph $C_4 - C_5(-C_6-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 122 cases of these 126 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_6-)$ in $K(\alpha, \beta)$. These cases are listed in table 10. In the following, it can be seen that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_6-)$.

- (1) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$:
 $R_1 \Rightarrow h_2^3 h_3^{-1} = h_3 \Rightarrow R_3 : h_3 h_2^{-1} h_3^{-1} h_2 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2(h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-1} h_2(h_3 h_2^{-1})^3 h_3 = 1$:
 $R_1 \Rightarrow (h_3 h_2^{-1})^2 h_3 h_2 = 1 \Rightarrow R_3 : h_3^{-1} h_2 h_3 h_2^{-1} = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (3) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-4} h_2 h_3 = 1$:
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (4) $R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3^{-1})^3 h_2 h_3 h_2^{-1} h_3 = 1$:
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

TABLE 10. All the relations related to the existence of a $C_4 - C_5(-C_6 -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$
2	$h_2(h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2(h_3 h_2^{-1})^3 h_3 = 1$
3	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 h_3 = 1$
4	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 h_2^{-1} h_3 = 1$

Before finding other forbidden subgraphs, we give a useful lemma.

Lemma 8.1. *Suppose that G is a group with the generating set S and H is a group with the generating set T . Also suppose that $H \leq G$. If for every $g \in S \cup S^{-1}$ and $h \in T$, $h^g = g^{-1} h g \in T$, then $H \trianglelefteq G$.*

Proof. The proof is straightforward. \square

Corollary 8.2. *Suppose that $G = \langle x, y \rangle$ and $y^2 \in Z(G)$ and $H = \langle x, x^y \rangle$. Then $H \trianglelefteq G$.*

8.7. $C_4 - C_5(-C_6 -)$. It can be seen that there are 462 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_6 in the graph $C_4 - C_5(-C_6 -)$. By considering all groups with two generators h_2 and h_3 and two relations which are between these cases and by using GAP [9], we see that 436 groups are finite and solvable, or just finite. So, there are just 22 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_6 -)$ in $K(\alpha, \beta)$. These cases are listed in table 11. In the following, it can be seen that all of these 22 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_6 -)$.

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1$:
 $R_1 \Rightarrow h_3 h_2^2 = h_2 h_3^{-1} (*) \Rightarrow R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$.
 $R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (2) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2(h_3 h_2^{-1})^2 h_3 = 1$:
 $R_1 \Rightarrow h_3 h_2^2 = h_2 h_3^{-1} (*) \Rightarrow R_3 : h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (3) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1$, $R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^3 h_3^2 = 1 \Rightarrow h_3 h_2^2 = h_3^{-1} h_2^{-1} (*)$, and $R_3 : h_3 h_2^2 h_3 h_2 h_3^{-1} h_2^{-1} h_3 h_2 = 1$. By $(*)$ and R_3 we have $(h_3^{-1} h_2^{-1} h_3 h_2)^2 = 1$ and G is torsion-free $\Rightarrow h_3^{-1} h_2^{-1} h_3 h_2 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (4) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1$, $R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $R_1 \Rightarrow h_2^2 h_3 h_2^{-1} h_3 = 1 (*)$ and $R_3 : h_2^2 h_3 h_2^{-1} h_3 h_2^{-1} h_3^{-2} = 1 \Rightarrow h_2 = h_3^{-2} \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2^2 h_3^{-3} h_2 h_3 = 1$:
 $R_1 \Rightarrow h_2^2 h_3^{-2} = h_2^{-1} (*) \Rightarrow R_3 : h_2^{-1} h_3^{-1} h_2 h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1$:
 $R_1 \Rightarrow h_3^2 \in Z(G)$, where $G = \langle h_2, h_3 \rangle$, and $h_2^2 h_3^{-1} = h_2^{-1} h_3 (*)$, and $R_3 : h_2^2 h_3^{-1} h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$. By $(*)$ and R_3 we have $(h_2^{-1} h_3)^2 (h_2 h_3^{-1})^2 = 1 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2 (**)$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$. If $H = \langle x, x^{h_3} \rangle$, by $(**)$ we have $H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \leq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (7) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $R_1 \Rightarrow h_3^{-2} h_2^2 = h_2^{-1} (*)$ and $R_3 : h_3 h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2^2 = 1$. By $(*)$ and R_3 we have $h_2 = (h_3 h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.
- (8) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $R_1 \Rightarrow h_2^2 h_3^{-1} = h_2^{-1} h_3 (*)$, and $R_3 : h_2 h_3^{-1} h_2^2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$. By $(*)$ and R_3 we have $(h_2 h_3^{-1} h_2^{-1} h_3)^2 = 1$ and G is torsion-free $\Rightarrow h_2 h_3^{-1} h_2^{-1} h_3 = 1 \Rightarrow h_2^{-1} h_3 h_2 = h_3 \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (9) $R_1 : h_2^2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^4 h_2 = 1$:
 $R_1 \Rightarrow h_2^3 h_3^{-1} = h_3 (*) \Rightarrow R_3 : h_2 h_3^{-1} h_2 h_3^{-1} h_2 = 1 \Rightarrow h_2 = (h_3 h_2^{-1})^2 \Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.
- (10) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$, $R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1$:
 $R_1 \Rightarrow (h_2^{-1} h_3)^2 = h_3^{-1} h_2^{-1} (*)$. By $(*)$ and R_3 we have $h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 (**)$. Using Tietze transformation where $h_2 \mapsto h_2 h_3$ and $h_3 \mapsto h_3$, we have $R_2 : (h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2 (***)$ and $(**)$: $h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. $(***) \Rightarrow h_3^2 \in Z(G)$, where $G = \langle h_2, h_3 \rangle$. Let $x = h_2^{-1} h_3^{-1}$. So $h_3^{-1} h_2^{-1} = x^{h_3}$. If $H = \langle x, x^{h_3} \rangle$, by $(***)$ we have $H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \leq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (11) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$, $R_3 : h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$:
 $R_1 \Rightarrow (h_3 h_2^{-1})^2 h_3 = h_2^{-1} (*) \Rightarrow R_3 : h_3^3 h_2^{-1} = 1 \Rightarrow h_2 = h_3^3 \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (12) $R_1 : h_2 h_3^{-3} h_2 = 1$, $R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$, $R_3 : (h_2 h_3^{-1})^3 h_3^{-1} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

TABLE 11. The relations of a $C_4 - C_5(-C_6-)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
3	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$
4	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
5	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-3} h_2 h_3 = 1$
6	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1$
7	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
8	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1$
9	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_2 h_3^{-1})^4 h_2 = 1$
10	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1$
11	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$
12	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^3 h_3^{-1} h_2^{-1} h_3 = 1$
13	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_3 (h_3 h_2^{-1})^4 h_3 = 1$
14	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1$
15	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
16	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
17	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
18	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$
19	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
20	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1$
21	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 (h_2 h_3^{-1})^2 h_2 = 1$
22	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$

(13) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_3 (h_3 h_2^{-1})^4 h_3 = 1:$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

(14) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1:$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

(15) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

(16) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

(17) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

(18) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

(19) $R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(20) \ R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(21) \ R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^3 (h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(22) \ R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

8.8. $\mathbf{C_4 - C_5(-C_7 -)}$. With the same discussion such as about C_4 cycles and considering the relations of C_7 cycles equivalent, there are 1173 non-equivalent cases for the relations of a C_7 cycle in the graph $K(\alpha, \beta)$. By considering these relations, it can be seen that there are 648 cases for the relations of a cycle C_4 , a cycle C_5 and a cycle C_7 in the graph $C_4 - C_5(-C_7 -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 608 cases of these 648 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 40 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_5(-C_7 -)$ in $K(\alpha, \beta)$. These cases are listed in table 12. In the following, we show that all of these 40 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_5(-C_7 -)$.

$$(1) \ R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : h_2^3 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(2) \ R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : h_2^2 (h_2 h_3^{-1})^4 h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(3) \ R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : h_2 h_3^{-2} h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(4) \ R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : (h_2 h_3^{-1})^4 h_3^{-1} h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(5) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^3 h_3^4 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(6) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^3 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(7) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2 h_3^2 h_2^{-1} h_3^4 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(8) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(9) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^3 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-1} \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.

$$(10) \ R_1 : h_2^2 h_3^{-2} h_2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

Table 12: The relations of a $C_4 - C_5(-C_7 - -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^3 h_3^2 h_2^{-1} h_3^2 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 (h_2 h_3^{-1})^4 h_2 = 1$
3	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 h_2^{-1} h_3^2 = 1$
4	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^4 h_3^{-1} h_2 h_3^{-1} h_2 = 1$
5	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^4 = 1$
6	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
7	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^4 = 1$
8	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
9	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-1} (h_3^{-1} h_2)^3 = 1$
10	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 (h_3^{-1} h_2^{-1})^2 h_3^2 = 1$
11	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$
12	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
13	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
14	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^3 h_3 h_2^{-1} h_3 = 1$
15	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
16	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$
17	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 h_2 h_3^{-1} h_2 = 1$
18	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$
19	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$
20	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$
21	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2^{-2} h_3 = 1$
22	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
23	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1$
24	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
25	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$
26	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} (h_3^{-1} h_2)^3 = 1$
27	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^4 h_3^3 = 1$
28	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^3 h_3 h_2^2 h_3^{-1} h_2 = 1$
29	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3 (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1$
30	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
31	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3^3 = 1$
32	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 h_2^{-2} h_3 = 1$
33	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^4 h_3^{-1} h_2 h_3^{-1} h_2 = 1$
34	$h_2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_3^2 (h_3 h_2^{-1})^4 h_3 = 1$
35	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$
36	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 (h_2 h_3^{-1})^3 h_2 h_3 = 1$
37	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$
38	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3^2 = 1$
39	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
40	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$

- (11) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (12) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1$:
 $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$.
 R_1 and $R_3 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (13) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
 $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2^{-1} h_3^{-1}$. So $h_3^{-1} h_2^{-1} = x^{h_3}$.
 R_1 and $R_3 \Rightarrow (h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (14) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 h_3 h_2^{-1} h_3 = 1$:
 $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1 \Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$.
 R_1 and $R_3 \Rightarrow (h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (15) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
 $R_2 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3^{-1} h_2^{-1}$. So $h_2^{-1} h_3^{-1} = x^{h_2}$.
 R_1 and $R_3 \Rightarrow (h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (16) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (17) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (18) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (19) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$:
 $R_2 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$.
 R_1 and $R_3 \Rightarrow (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (20) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$:
 $R_2 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$.
 R_1 and $R_3 \Rightarrow (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (21) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(22) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(23) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(24) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(25) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(26) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-3} (h_3^{-1} h_2)^3 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(27) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2^4 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(28) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2^3 h_3 h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(29) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(30) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(31) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(32) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(33) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : (h_2 h_3^{-1})^4 h_3^{-1} h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(34) R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 (h_3 h_2^{-1})^4 h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(35) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(36) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(37) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(38) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(39) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(40) R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

C₅ – C₅ subgraph: Suppose that there are two cycles of length 5 in the graph $K(\alpha, \beta)$. By considering the relations from Table 4 which are not disproved, it can be seen that there are 2485

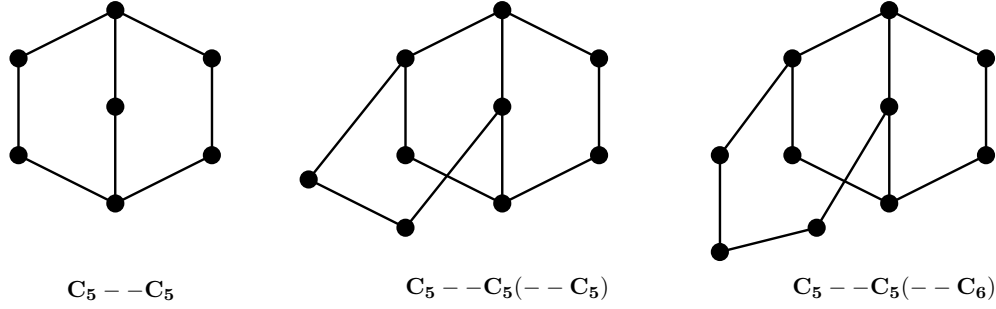


FIGURE 16. The graph $C_5 - - C_5$ and some forbidden subgraphs which contain it

cases for existing two cycles C_5 in the graph $K(\alpha, \beta)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and two relations which are between 2038 cases of these 2485 cases are solvable or have the same “structure description” $SL(2, 5)$ according to the function StructureDescription of GAP, that is finite. So there are 447 cases for the relations of two cycles of length 5 in the graph $K(\alpha, \beta)$. Now suppose that the graph $K(\alpha, \beta)$ has a subgraph isomorphic to the graph $C_5 - - C_5$. Since this structure has two cycles of length 5, the relations of these C_5 cycles must be between 447 cases that have mentioned above.

Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5]$ and $[h'_1, h''_1, h'_2, h''_2, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8]$ are 10-tuples related to the cycles C_5 in the graph $C_5 - - C_5$, where the first four components of these tuples are related to the common edges of C_5 and C_5 . Without loss of generality we may assume that $h'_1 = 1$, where $h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. With the same discussion such as about $K_{2,3}$, it is easy to see that $h'_3 \neq h'_6$ and $h''_5 \neq h''_8$.

With such assumptions and by the discussion above, it can be seen that there are 99 cases which may lead to the existence of a subgraph isomorphic to the graph $C_5 - - C_5$ in the graph $K(\alpha, \beta)$. These cases are listed in table 13. In the following, we show that 87 cases of these relations lead to a contradiction and just 12 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_5 - - C_5$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 13.

- (1) $R_1 : h_2^3 h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (4) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (5) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.

Table 13: The relations of a $C_5 - -C_5$

n	R_1	R_2	n	R_1	R_2
1	$h_2^3 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	51	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
2	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	52	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
3	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	53	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
4	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	54	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
5	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	55	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
6	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_2^2 h_3^{-1} h_3^2 = 1$	56	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
7	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	57	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
8	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	58	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$
9	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	59	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
10	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	60	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
11	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	61	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
12	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	62	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
13	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	63	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$
14	$h_2^2 h_3^3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	64	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
15	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	65	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$
16	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	66	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
17	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	67*	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
18	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	68	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$
19	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	69	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
20	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	70	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$
21*	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-3} h_2 = 1$	71	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$
22	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	72	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-4} h_2 = 1$
23*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	73	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$
24	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	74	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
25	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	75	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
26	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	76	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
27	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	77	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
28	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	78	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$
29	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	79	$h_2 h_3 h_2^{-2} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
30	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	80*	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
31	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	81	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$
32*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	82	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$
33	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	83*	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
34	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	84	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
35*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	85	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
36	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	86	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$
37*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	87	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-4} h_2 = 1$
38*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	88	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
39	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	89	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
40	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	90	$h_2 h_3^{-4} h_2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
41	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	91	$h_2 h_3^{-4} h_2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
42	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	92	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$
43*	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	93	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
44	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	94*	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
45	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	95	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
46*	$h_2^2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	96	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
47	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	97	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
48	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	98	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
49	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	99	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
50	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$			

(6) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$

$\Rightarrow h_3 = 1$ that is a contradiction.

(7) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$

$\Rightarrow h_2 = 1$ that is a contradiction.

- (8) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (9) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (10) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (11) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (12) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (13) $R_1 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_3 h_2^{-2} \rangle = \langle h_3 h_2^{-2} \rangle$ is abelian, a contradiction.
- (14) $R_1 : h_2^2 h_3^3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (15) $R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (16) $R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (17) $R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (18) $R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (19) $R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (20) $R_1 : h_2^2 h_3 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (22) $R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (24) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (25) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (26) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (27) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (28) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have: $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3$. So $h_2^{-1} h_3^{-1} = (x^{h_2})^{-1}$.

$R_2 \Rightarrow (h_2 h_3)^2 = (h_2^{-1} h_3^{-1})^2$ so $H = \langle x, (x^{h_2})^{-1} \rangle = \langle x, x^{h_2} \rangle \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(29) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(30) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have: $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$.

$R_2 \Rightarrow (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(31) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(33) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(34) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(36) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(39) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(40) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(41) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (31) and with the same discussion, there is a contradiction.

$$(42) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(44) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2, h_2^{-1} h_3 \rangle = \langle h_2^{-1} h_3 \rangle$ is abelian, a contradiction.

$$(45) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(47) \quad R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(48) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(49) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(50) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.

$$(51) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(52) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(53) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(54) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. So by the same discussion such as item 50, there is a contradiction.

$$(55) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(56) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(57) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(58) \quad R_1 : h_2^2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(59) \quad R_1 : h_2^2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(60) \quad R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(61) \quad R_1 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(62) \quad R_1 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$$(63) \quad R_1 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1} h_2)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(64) \quad R_1 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.

$$(65) \quad R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, \quad R_2 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(66) \quad R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, \quad R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (48) and with the same discussion, there is a contradiction.

$$(68) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1, \quad R_2 : (h_2 h_3^{-1} h_2)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(69) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(70) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (53) and with the same discussion, there is a contradiction.

$$(71) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (57) and with the same discussion, there is a contradiction.

$$(72) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(73) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(74) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(75) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(76) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (54) and with the same discussion, there is a contradiction.

$$(77) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (49) and with the same discussion, there is a contradiction.

$$(78) R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (63) and with the same discussion, there is a contradiction.

$$(79) R_1 : h_2 h_3 h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(81) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.

$$(82) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (68) and with the same discussion, there is a contradiction.

$$(84) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(85) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(86) R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(87) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(88) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(89) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(90) R_1 : h_2 h_3^{-4} h_2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(91) R_1 : h_2 h_3^{-4} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(92) R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(93) \ R_1 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (61) and with the same discussion, there is a contradiction.

$$(95) \ R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, \ R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (84) and with the same discussion, there is a contradiction.

$$(96) \ R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, \ R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(97) \ R_1 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (60) and with the same discussion, there is a contradiction.

$$(98) \ R_1 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, \ R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(99) \ R_1 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \ R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (98) and with the same discussion, there is a contradiction.

8.9. $C_5 - -C_5(- - C_5)$. It can be seen that there are 192 cases for the relations of three cycles C_5 in the graph $C_5 - -C_5(- - C_5)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 188 cases of these 192 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - -C_5(- - C_5)$ in $K(\alpha, \beta)$. These cases are listed in table 14. In the following, we show that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - -C_5(- - C_5)$.

$$(1) \ R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \ R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \ R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(2) \ R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(3) \ R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(4) \ R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \ R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

TABLE 14. The relations of a $C_5 - -C_5(- - C_5)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
2	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
3	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
4	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$

8.10. $C_5 - -C_5(- - C_6)$. It can be seen that there are 1006 cases for the relations of two cycles C_5 and a cycle C_6 in the graph $C_5 - -C_5(- - C_6)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 986 cases of these 1006 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 20 cases for the relations of these cycles which

may lead to the existence of a subgraph isomorphic to the graph $C_5 - -C_5(- - C_6)$ in $K(\alpha, \beta)$. These cases are listed in table 15. In the following, we show that all of these 20 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - -C_5(- - C_6)$.

TABLE 15. The relations of a $C_5 - -C_5(- - C_6)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
2	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
3	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$
4	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2^2 h_3^{-3} h_2 h_3 = 1$
5	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3 h_2 h_3^{-3} h_2 = 1$
6	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
7	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
8	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
9	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-4} h_2 h_3 = 1$
10	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1$
11	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1$
12	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
13	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-2} h_3^2 = 1$
14	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-2} h_3^3 = 1$
15	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
16	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
17	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3 h_2 h_3^{-2} h_2 = 1$
18	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-2} h_2^2 h_3 = 1$
19	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
20	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$

- (1) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (4) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (5) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-3} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(7) R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(8) R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(9) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-4} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(10) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(11) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(12) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(13) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^2 h_2^{-2} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(14) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3^3 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have:

$$R_2 \Rightarrow h_3^2 \in Z(G) \text{ where } G = \langle h_2, h_3 \rangle. \text{ Let } x = h_2^{-1} h_3^{-1}. \text{ So } h_3^{-1} h_2^{-1} = x^{h_3}.$$

R_1 and $R_3 \Rightarrow (h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(15) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(16) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

$\Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$. Furthermore we have $(h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(17) R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3 h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(18) R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-2} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(19) R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(20) R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$C_4 - C_6$ subgraph: Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4]$ is the 8-tuple related to the cycle C_4 and $[h'_1, h''_1, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8, h'_9, h''_9]$ is the 12-tuple related to the cycle C_6 in the graph $C_4 - C_6$, where the first two components of these tuples are related to the common edge of C_4 and C_6 . With the same argument such as about $C_4 - C_5$, without loss of generality we may assume that $h'_1 = 1$, where $h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8, h'_9, h''_9 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. Also it

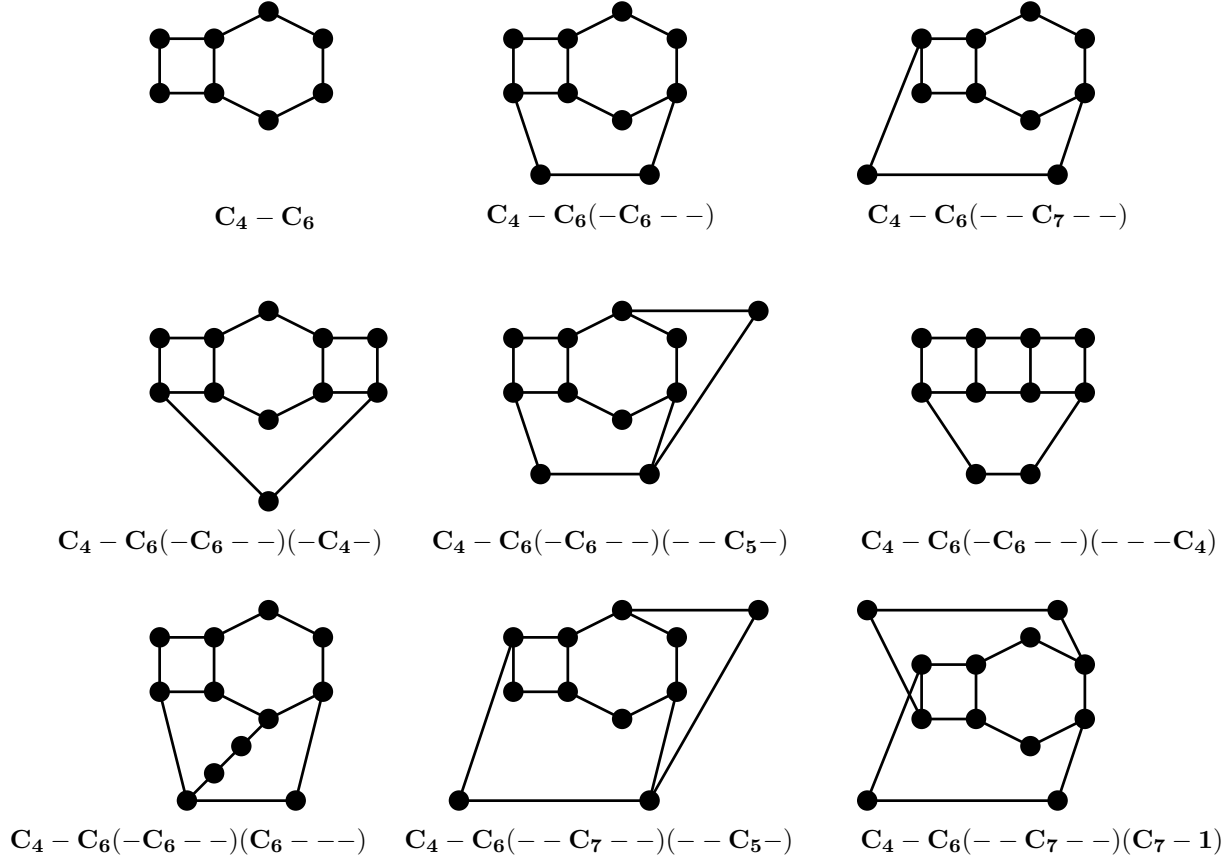


FIGURE 17. The graphs $C_4 - C_6$, $C_4 - C_6(-C_6 - -)$ and $C_4 - C_6(- - C_7 - -)$, and some forbidden subgraphs which contain them

is easy to see that $h'_2 \neq h'_5$ and $h''_4 \neq h''_9$. With these assumptions and by considering the relations from Tables 2 and 5 which are not disproved, it can be seen that there are 2188 cases for the relations of the cycles C_4 and C_6 in this structure. Using Gap [9], we see that all groups with two generators h_2 and h_3 and two relations which are between 2035 cases of these 2188 cases are finite and solvable and 14 groups have the same “structure description” $SL(2, 5)$ according to the function StructureDescription of GAP, that is finite. So there are just 139 cases for the relations of the cycles C_4 and C_6 which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6$ in the graph $K(\alpha, \beta)$. These cases are listed in table 16. In the following, we show that 127 cases of these relations lead to a contradiction and just 12 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 16.

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1$, $R_2 : h_2^3 h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

Table 16: The relations of a $C_4 - C_6$ in $K(\alpha, \beta)$

n	R_1	R_2	n	R_1	R_2
1	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3 h_2^{-2} h_3 = 1$	40	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
2	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1$	41	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$
3	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	42	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 (h_2 h_3^{-1})^2 h_2 = 1$
4	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	43	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
5	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	44	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2 h_3 h_2^{-1} h_3 = 1$
6	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	45	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1$
7	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^4 = 1$	46	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$
8	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$	47	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
9	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1$	48*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$
10	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$	49*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
11	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_2^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$	50	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
12	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$	51	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
13	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$	52	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
14	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3 h_2 h_3^{-1} h_2 = 1$	53	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
15	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$	54	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$
16	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2 h_3^{-3} h_2 = 1$	55	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
17	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^2 h_2 h_3^{-2} h_2 = 1$	56	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1$
18	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$	57	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
19	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$	58	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
20	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$	59	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1$
21	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	60	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1$
22	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1$	61	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1$
23	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$	62	$h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$
24	$h_2^2 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1$	63	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^2 h_2^{-2} h_3 = 1$
25	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	64	$h_2 h_3^{-3} h_2 = 1$	$h_3^2 h_3 h_2^{-2} h_3^2 = 1$
26	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	65	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1$
27	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-3} h_2 h_3 = 1$	66	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1$
28	$h_2^2 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-2} h_2 h_3^2 = 1$	67	$h_2 h_3^{-3} h_2 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
29	$h_2^2 h_3^{-2} h_2 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1$	68	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
30	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-3} h_3 = 1$	69	$h_2 h_3^{-3} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$
31	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^3 = 1$	70	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
32	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	71	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
33	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$	72	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
34*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	73	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1$
35	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	74	$h_2 h_3^{-3} h_2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
36	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1$	75	$h_2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-4} h_2 h_3 = 1$
37	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	76	$h_2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^3 h_3^{-1} h_2^{-1} h_3 = 1$
38	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$	77	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^3 h_3^{-2} h_2^{-1} h_3 = 1$
39*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	78*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^3 h_2 = 1$

Continued on next page

Table 16 – continued from previous page

n	R_1	R_2	n	R_1	R_2
79	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3h_2h_3^{-1}h_2h_3 = 1$	110	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2^2h_3^{-2}(h_2^{-1}h_3)^2 = 1$
80	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3h_2^{-1}h_3^{-2}h_2 = 1$	111	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3^{-2}h_2h_3^3 = 1$
81*	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3^{-1}(h_2h_3^{-1}h_2)^2 = 1$	112	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3^{-1}h_2h_3(h_3h_2^{-1})^2h_3 = 1$
82	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3h_2h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1$	113	$h_2h_3^{-1}h_2h_3^2 = 1$	$(h_2h_3^{-1})^2h_2h_3^2h_2^{-1}h_3 = 1$
83	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3h_2h_3^{-1}(h_3^{-1}h_2)^2 = 1$	114*	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^3h_3^3 = 1$
84*	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3(h_2h_3^{-1})^2h_3^{-1}h_2 = 1$	115	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^3h_3^{-1}(h_2^{-1}h_3)^2 = 1$
85	$h_2h_3^{-2}h_2h_3 = 1$	$h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1$	116*	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2(h_2h_3)^2h_3 = 1$
86	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3^{-2}(h_2^{-1}h_3)^2 = 1$	117*	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^2h_3^2h_2h_3 = 1$
87*	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3^{-1}(h_3^{-1}h_2)^2h_3 = 1$	118	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1$
88	$h_2h_3^{-2}h_2h_3 = 1$	$h_2(h_2h_3^{-1})^2h_3^{-1}h_2h_3 = 1$	119	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^2h_3^{-1}h_2h_3h_2h_3^{-1}h_2 = 1$
89	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$	120	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3^2h_2^{-1}h_3h_2h_3^{-1}h_2 = 1$
90	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-2}(h_2^{-1}h_3)^3 = 1$	121	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3h_2^{-2}h_3h_2h_3^{-1}h_2 = 1$
91	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-3}h_2h_3^2 = 1$	122*	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1$
92	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-2}h_2^2h_3^{-1}h_2h_3 = 1$	123	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1$
93	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-2}h_2h_3h_2h_3^{-1}h_2 = 1$	124	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3^2 = 1$
94	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3h_2^{-1}h_3 = 1$	125	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3h_2^{-1}h_3^3h_2^{-1}h_3 = 1$
95	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3^{-2}h_2 = 1$	126	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3^{-3}(h_2^{-1}h_3)^2 = 1$
96	$h_2h_3^{-2}h_2h_3 = 1$	$(h_2h_3^{-1})^2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	127	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$h_2h_3^{-2}h_2h_3h_2^{-1}h_3^2 = 1$
97	$h_2h_3^{-2}h_2h_3 = 1$	$h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2h_3 = 1$	128	$h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1$	$(h_2h_3^{-1})^2h_2^{-1}h_3^3 = 1$
98	$h_2h_3^{-2}h_2h_3 = 1$	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2h_3 = 1$	129	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2^3(h_2h_3^{-1})^2h_2 = 1$
99	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2^4h_3^2 = 1$	130	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2^2(h_2h_3^{-1})^2h_2^{-1}h_3 = 1$
100	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2^3h_3^{-1}h_2^{-1}h_3^2 = 1$	131	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2^3h_3h_2^{-1}(h_3^{-1}h_2)^2 = 1$
101	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2^2h_3^2h_2^{-1}h_3^{-1}h_2 = 1$	132	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2h_3h_2^{-1}h_3^{-1}h_2^2h_3^{-1}h_2 = 1$
102	$h_2h_3^{-1}h_2h_3^2 = 1$	$(h_2h_3)^2h_2^{-1}h_3^{-1}h_2 = 1$	133	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1$
103	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1$	134	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2^2h_3^{-2}(h_2^{-1}h_3)^2 = 1$
104	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2^2h_3^{-1}h_2^{-1}h_3h_2h_3 = 1$	135	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$
105	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$	136	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2h_3^{-2}(h_3^{-1}h_2)^2h_3 = 1$
106	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$	137	$(h_2h_3^{-1})^2h_2h_3 = 1$	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3^2 = 1$
107	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2h_3 = 1$	138	$(h_2h_3^{-1})^2h_2h_3 = 1$	$(h_2h_3^{-1})^2h_3^{-2}h_2h_3 = 1$
108	$h_2h_3^{-1}h_2h_3^2 = 1$	$h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1$	139	$(h_2h_3^{-1})^2h_2h_3 = 1$	$(h_2h_3^{-1})^2h_3^{-1}h_2h_3^2 = 1$
109	$h_2h_3^{-1}h_2h_3^2 = 1$	$(h_2h_3^{-1})^2h_2h_3h_2^{-1}h_3^2 = 1$			

- (2) $R_1 : h_2^2h_3h_2^{-1}h_3 = 1$, $R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (3) $R_1 : h_2^2h_3h_2^{-1}h_3 = 1$, $R_2 : h_2^2h_3^{-2}(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (4) $R_1 : h_2^2h_3h_2^{-1}h_3 = 1$, $R_2 : h_2^2h_3^{-1}h_2(h_3h_2^{-1})^2h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (5) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (7) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^4 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (8) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (10) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (11) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (12) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (13) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (14) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (15) $R_1 : h_2^2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (16) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-3} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (17) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (18) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (19) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$:
 $\Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$. Also $(h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (20) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.
- (21) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.
- (22) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (23) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$:
 With the same discussion such as the item 19, there is a contradiction.

- (24) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (25) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (26) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2^{-1} h_3 \rangle$ is abelian, a contradiction.
- (27) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-3} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (28) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2 h_3^2 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (29) $R_1 : h_2^2 h_3^{-2} h_2 = 1, R_2 : h_2 (h_2 h_3^{-1})^3 h_2^{-1} h_3 = 1:$
 With the same discussion such as the item 19, there is a contradiction.
- (30) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-3} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (31) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (32) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (33) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (35) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (36) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (37) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (38) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (40) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (41) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (42) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2 (h_2 h_3^{-1})^2 h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (43) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (44) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2 h_3 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (45) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (46) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (47) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (50) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (51) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (52) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (53) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (54) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (55) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (56) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (57) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (58) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (59) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (60) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (61) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (62) $R_1 : h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (63) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^2 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.
- (64) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.
- (65) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1$:
 By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.
- (66) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3^{-1} h_2 = 1$:
 By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.
- (67) $R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$:
 By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(68) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(69) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(70) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(71) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(72) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(73) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(74) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(75) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-4} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.

$$(76) R_1 : h_2 h_3^{-3} h_2 = 1, R_2 : (h_2 h_3^{-1})^3 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$$(77) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (33) and with the same discussion, there is a contradiction.

$$(79) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3 h_2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (44) and with the same discussion, there is a contradiction.

$$(80) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(82) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(83) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.

$$(85) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(86) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (32) and with the same discussion, there is a contradiction.

$$(88) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(89) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (40) and with the same discussion, there is a contradiction.

$$(90) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (31) and with the same discussion, there is a contradiction.

$$(91) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (30) and with the same discussion, there is a contradiction.

$$(92) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(93) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.

$$(94) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (41) and with the same discussion, there is a contradiction.

$$(95) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.

$$(96) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (43) and with the same discussion, there is a contradiction.

$$(97) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (50) and with the same discussion, there is a contradiction.

$$(98) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (51) and with the same discussion, there is a contradiction.

$$(99) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2^4 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(100) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2^3 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(101) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2^2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(102) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(103) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(104) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(105) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(106) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(107) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(108) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(109) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(110) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(111) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(112) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(113) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(115) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2^3 h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(118) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2^2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(119) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(120) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(121) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(123) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(124) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (120) and with the same discussion, there is a contradiction.

$$(125) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (119) and with the same discussion, there is a contradiction.

$$(126) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-3} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (115) and with the same discussion, there is a contradiction.

$$(127) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (121) and with the same discussion, there is a contradiction.

$$(128) \quad R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (118) and with the same discussion, there is a contradiction.

$$(129) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2^3 (h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (62) and with the same discussion, there is a contradiction.

$$(130) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2^2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (61) and with the same discussion, there is a contradiction.

$$(131) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2^2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (60) and with the same discussion, there is a contradiction.

$$(132) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (59) and with the same discussion, there is a contradiction.

$$(133) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (58) and with the same discussion, there is a contradiction.

$$(134) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (55) and with the same discussion, there is a contradiction.

$$(135) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (57) and with the same discussion, there is a contradiction.

$$(136) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (54) and with the same discussion, there is a contradiction.

$$(137) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(138) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-2} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(139) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (53) and with the same discussion, there is a contradiction.

$C_4 - C_6(-C_6 - -)$ subgraph: By considering the relations from Tables 5 and 16 which are not disproved, it can be seen that there are 342 cases for the relations of a cycle C_4 and two cycles C_6 in the graph $C_4 - C_6(-C_6 - -)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 324 cases of these 342 cases are finite and solvable, that is a contradiction with the assumptions. So there are just 18 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)$ in the graph $K(\alpha, \beta)$. These cases are listed in table 17. In the following, we show that 12 cases of these relations lead to a contradiction and just 6 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 17.

TABLE 17. The relations of a $C_4 - C_6(-C_6 - -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
2	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$
3	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$
4*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$
5	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
6	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^3 h_2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1$
7*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$
8	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1$
9	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
10*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$
11	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
12	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^3 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1$
13	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
14*	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$
15	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1$
16*	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$
17	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1$
18	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1$

$$(2) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(3) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(5) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(6) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(8) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(9) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(11) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^3 h_3^3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(12) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^3 h_3^3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(13) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3)^2 h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(15) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3)^2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(17) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^2 h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(18) R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^2 h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$C_4 - C_6(- - C_7 - -)$ subgraph: With the same discussion such as about C_4 cycles, by considering the relations of C_7 cycles and the relations from Table 16 which are not disproved, it can be seen that there are 521 cases for the relations of a cycle C_4 , a cycle C_6 and a cycle C_7 in the graph $C_4 - C_6(- - C_7 - -)$. By considering all groups with two generators h_2 and h_3 and three relations which are between these cases and by using Gap [9], we see that 484 groups are finite and solvable and 4 groups have the same “structure description” $C_5 \times SL(2, 5)$ according to the function StructureDescription of GAP, that is finite. So there are just 33 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$. These cases are listed in table 18. In the following, we show that 17 cases of these relations lead to a contradiction and 16 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 18.

$$(7) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^3 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(10) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

TABLE 18. The relations of a $C_4 - C_6(- - C_7 - -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-5} h_2 h_3 = 1$
2*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1$
3*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1$
4*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^4 h_3 = 1$
5*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-2})^2 h_2^{-1} h_3^2 = 1$
6*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1$
7	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^3 h_3 = 1$
8*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 h_2^{-1} h_3 = 1$
9*	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
10	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1$
11	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3 (h_3 h_2^{-1})^2 h_3 = 1$
12	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$
13	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^3 h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1$
14	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1$
15	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_2^{-2} h_3 = 1$
16	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^3 h_2^{-1} h_3^3 = 1$
17*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$(h_2^2 h_3^{-1})^2 h_2^{-2} h_3 = 1$
18*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1} h_2)^2 = 1$
19*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
20*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1$
21*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2^4 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$
22*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
23*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1$
24*	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^4 h_2 h_3 h_2^{-1} h_3 = 1$
25	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-3} h_2^{-1} h_3 = 1$
26	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^3 = 1$	$h_2^3 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
27	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 h_3 = 1$
28	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3^2 = 1$
29	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
30	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$
31	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$h_2^2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1$
32	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 h_2 h_3 = 1$
33	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$	$h_2 ((h_3^{-1} h_2)^2 h_3^{-1})^2 h_2 = 1$

(11) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_3 h_2^{-1} h_3)^2 = 1$, $R_3 : (h_2 h_3^{-1})^2 h_2^{-1} h_3 (h_3 h_2^{-1})^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

- (12) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (13) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1, R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.
- (14) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1:$
 By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.
- (15) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_2^{-2} h_3 = 1:$
 By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.
- (16) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^3 = 1:$
 By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.
- (25) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^3 h_3^3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-3} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (26) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^3 h_3^3 = 1, R_3 : h_2^3 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$
 By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.
- (27) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3)^2 h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (28) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3)^2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (29) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3)^2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (30) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^2 h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.
- (31) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^2 h_2 h_3 = 1, R_3 : h_2^2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1:$
 By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.
- (32) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^2 h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 h_2 h_3 = 1:$
 By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.
- (33) $R_1 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1, R_3 : h_2 ((h_3^{-1} h_2)^2 h_3^{-1})^2 h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

8.11. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(\mathbf{C}_7 - 1)$. By considering the relations of a C_7 cycle in the graph $K(\alpha, \beta)$ and relations from Table 18 which are not disproved, it can be seen that there are 176 cases for the relations of a cycle C_4 , two cycles C_7 and a cycle C_6 in this structure. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 1)$.

8.12. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(- - \mathbf{C}_5 -)$. By considering the relations from Tables 4 and 18 which are not disproved, it can be seen that there are 28 cases for the relations of a cycle C_4 , a cycle C_6 , a cycle C_7 and a cycle C_5 in the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 24 cases of these 28 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases

for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$ in $K(\alpha, \beta)$. These cases are listed in table 19. In the following, we show that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - C_5 -)$.

TABLE 19. The relations of a $C_4 - C_6(- - C_7 - -)(- - C_5 -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3	R_4
1	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
2	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$
3	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
4	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$

$$(1) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1,$$

$$R_4 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(2) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1,$$

$$R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(3) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1,$$

$$R_4 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(4) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1,$$

$$R_4 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

8.13. $\mathbf{C_4 - C_6(-C_6 - -)(-C_4 -)}$. By considering the relations from Tables 2 and 17 which are not disproved, it can be seen that there is no case for the relations of two cycles C_4 and two cycles C_6 in this structure. It means that the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(-C_4 -)$.

8.14. $\mathbf{C_4 - C_6(-C_6 - -)(- - C_5 -)}$. By considering the relations from Tables 4 and 17 which are not disproved, it can be seen that there are 22 cases for the relations of a cycle C_4 , two cycles C_6 and a cycle C_5 in this structure. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(- - C_5 -)$.

8.15. $\mathbf{C_4 - C_6(-C_6 - -)(C_6 - -)}$. By considering the relations from Tables 5 and 17 which are not disproved, it can be seen that there are 66 cases for the relations of a cycle C_4 and three cycles C_6 in the graph $C_4 - C_6(-C_6 - -)(C_6 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 62 cases of these 66 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(C_6 - -)$ in $K(\alpha, \beta)$.

These cases are listed in table 20. In the following, we show that all of these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(C_6 - - -)$.

$$(1) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1,$$

$$R_4 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(2) R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1,$$

$$R_4 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(3) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1,$$

$$R_4 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(4) R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1,$$

$$R_4 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

TABLE 20. The relations of a $C_4 - C_6(-C_6 - -)(C_6 - - -)$ in $K(\alpha, \beta)$

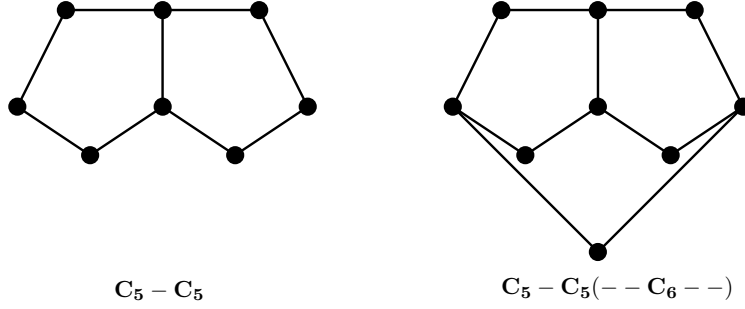
n	R_1	R_2	R_3	R_4
1	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1$
2	$h_2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$
3	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1$
4	$h_2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$	$h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$

8.16. **$C_4 - C_6(-C_6 - -)(- - - C_4)$.** By considering the relations from Tables 2 and 17 which are not disproved, it can be seen that there is no case for the relations of two cycles C_4 and two cycles C_6 in this structure. It means that the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_6 - -)(- - - C_4)$.

$C_5 - C_5$ subgraph: With the same discussion such as about $C_5 - -C_5$, there are 447 cases for the relations of the existence of two cycles of length 5 in the graph $K(\alpha, \beta)$. Now suppose that there are two C_5 with one common edge in the graph $K(\alpha, \beta)$. Since this structure has two cycles of length 5, the relations of these C_5 cycles must be between 447 cases that have mentioned above.

Suppose that $[h'_1, h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5]$ and $[h'_1, h''_1, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8, h'_9, h''_9]$ are 10-tuples related to the cycles C_5 in the graph $C_5 - C_5$, where the first two components of these tuples are related to the common edge of C_5 and C_5 . Without loss of generality we may assume that $h'_1 = 1$, where $h''_1, h'_2, h''_2, h'_3, h''_3, h'_4, h''_4, h'_5, h''_5, h'_6, h''_6, h'_7, h''_7, h'_8, h''_8, h'_9, h''_9 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. With the same discussion such as about $K_{2,3}$, it is easy to see that $h'_2 \neq h'_6$ and $h''_5 \neq h''_9$.

With these assumptions and by the discussion above, it can be seen that there are 355 cases which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5$ in the graph $K(\alpha, \beta)$. These cases are listed in table 21. In the following, we show that 225 cases of these relations lead to a contradiction and 130 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 21.

FIGURE 18. The graph $C_5 - C_5$ and a forbidden subgraph which contains it

- (1) $R_1 : h_2^3 h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^3 h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (4) $R_1 : h_2^3 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $R_1 \Rightarrow h_2^3 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$. Also $(h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (5) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (7) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (8) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (9) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.
- (10) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (11) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow h_2^3 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2^{-1} h_3^{-1}$. So $h_3^{-1} h_2^{-1} = x^{h_3}$. Also $(h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (14) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (15) $R_1 : h_2^3 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

- (16) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (17) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (18) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (19) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (20) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (21) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (23) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (24) $R_1 : h_2^3 h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $R_1 \Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3^{-1}$. So $h_3^{-1} h_2 = x^{h_3}$. Also $(h_2 h_3^{-1})^2 = (h_3^{-1} h_2)^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (26) $R_1 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-2} \rangle$ is abelian, a contradiction.
- (27) $R_1 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 (h_3 h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (28) $R_1 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2^{-1} h_3^{-1}$. So $h_3^{-1} h_2^{-1} = x^{h_3}$. Also $(h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (31) $R_1 : h_2^2 h_3^3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$
By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (32) $R_1 : h_2^2 h_3^3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (33) $R_1 : h_2^2 h_3^3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (35) $R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (36) $R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (37) $R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

Table 21: The relations of a $C_5 - C_5$

n	R_1	R_2	n	R_1	R_2
1	$h_2^3 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	61	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
2	$h_2^3 h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	62	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$
3*	$h_2^3 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	63*	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
4	$h_2^3 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	64*	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
5	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	65	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
6	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	66	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
7	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	67*	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
8	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	68*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$
9	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	69	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$
10	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	70	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$
11	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	71	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$
12*	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$	72	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$
13*	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	73	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-3} h_2 = 1$
14	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	74	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$
15	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	75	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$
16	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	76	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$
17	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	77	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$
18	$h_2^3 h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	78	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
19	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	79	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
20	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	80	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$
21	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	81	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$
22*	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	82*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$
23	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	83*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$
24	$h_2^3 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	84	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$
25*	$h_2^3 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	85	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$
26	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	86	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$
27	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	87	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$
28	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	88	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$
29*	$h_2^2 (h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	89	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
30*	$h_2^2 h_3^3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	90	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
31	$h_2^2 h_3^3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	91	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$
32	$h_2^2 h_3^3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	92	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-2} h_3^2 = 1$
33	$h_2^2 h_3^3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	93	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
34*	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2^{-1} h_3 = 1$	94	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
35	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	95	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$
36	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	96	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$
37	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	97	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
38	$h_2^2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	98*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$
39*	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	99	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
40	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	100	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$
41	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	101*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$
42	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	102*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$
43	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	103	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
44	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	104*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$
45	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	105	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
46	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	106	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
47	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	107*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$
48	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	108*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$
49	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	109*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
50*	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	110	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
51*	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	111	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$
52	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	112*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
53*	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	113	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$
54	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	114	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$
55	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	115*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
56	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	116*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
57	$h_2^2 h_3 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	117*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
58	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	118*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
59	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	119	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
60*	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2^2 h_3^{-3} h_2 = 1$	120*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$

Continued on next page

Table 21 – continued from previous page

n	R_1	R_2	n	R_1	R_2
121*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-2} h_3 = 1$	181	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
122*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$	182*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
123*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	183	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
124*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	184	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
125*	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	185	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
126	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	186*	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$
127	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$	187	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$
128	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$	188	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
129*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	189	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
130*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	190*	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
131	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-3} h_2 = 1$	191*	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
132	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	192*	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
133	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^2 = 1$	193	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
134	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	194*	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
135*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	195*	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
136	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	196	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
137*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	197*	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$
138*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	198	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
139	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	199	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 = 1$
140*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	200	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$
141*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	201	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
142*	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	202	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
143	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	203	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
144	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$	204*	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$
145*	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	205	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
146*	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	206*	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
147*	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	207	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$
148	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	208	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
149	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	209*	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
150*	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	210	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$
151	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-3} h_2 = 1$	211*	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$
152	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	212*	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
153	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	213	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$
154*	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	214	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
155*	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	215*	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$
156	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	216	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
157	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	217	$h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
158	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	218*	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$
159	$h_2^2 h_3^{-3} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	219	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-2} h_3^2 = 1$
160*	$h_2^2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	220	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
161*	$h_2^2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	221*	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
162*	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$	222	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
163	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 = 1$	223	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$
164	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	224*	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
165*	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	225*	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$
166	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	226	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$
167*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	227	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$
168	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	228*	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
169	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	229	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$
170*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	230	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-4} h_2 = 1$
171	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	231	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$
172	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	232	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
173*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	233	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$
174	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	234	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
175	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	235	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
176	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 = 1$	236*	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
177*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	237*	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
178	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$	238	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
179*	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	239	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$
180	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	240*	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$

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Table 21 – continued from previous page

n	R_1	R_2	n	R_1	R_2
241	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	299	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
242	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	300	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
243*	$h_2 h_3 h_2^{-2} h_3^2 = 1$	$h_2 h_3 h_2^{-2} h_3^2 = 1$	301	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
244	$h_2 h_3 h_2^{-2} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	302	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1$
245*	$h_2 h_3 h_2^{-2} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	303	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
246*	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	304*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
247	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	305	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
248*	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	306*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
249	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	307*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
250	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	308	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
251	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	309*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$
252*	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	310*	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
253*	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	311	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
254	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	312	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$
255*	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	313	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
256*	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	314	$h_2 h_3^{-4} h_2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
257*	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	315	$h_2 h_3^{-4} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
258*	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	316	$h_2 h_3^{-4} h_2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
259	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	317*	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$
260*	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	318	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 = 1$
261*	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	319*	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
262*	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	320	$h_2 h_3^{-3} h_2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
263*	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	321*	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
264	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	322	$h_2 h_3^{-3} h_2 h_3 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
265	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	323	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
266	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	324*	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$
267*	$h_2 (h_3 h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	325	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$
268*	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	326	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
269*	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	327*	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
270	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	328	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
271*	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	329	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
272	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	330*	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
273	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	331	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
274*	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	332*	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
275	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	333	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
276	$h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	334*	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
277*	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	335	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
278	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	336	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
279*	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	337	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
280*	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	338	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
281	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	339*	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
282*	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	340	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$
283	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	341*	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$
284	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	342*	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
285	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	343*	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
286*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	344*	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$
287	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	345	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$
288	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-4} h_2 = 1$	346	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
289	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	347	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
290	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$	348	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1$
291	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	349	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$
292	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	350*	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_3^2 (h_3 h_2^{-1})^2 h_3 = 1$
293	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	351	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$
294*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 = 1$	352*	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$
295	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$	353*	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
296	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	354*	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$
297	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$	355	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$
298*	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$			

$$(38) R_1 : h_2^2 h_3^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(40) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(41) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(42) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(43) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(44) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(45) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(46) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(47) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(48) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(49) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(52) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(54) R_1 : h_2^2 h_3 h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(55) R_1 : h_2^2 h_3 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(56) R_1 : h_2^2 h_3 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(57) R_1 : h_2^2 h_3 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(58) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.

$$(59) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(61) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(62) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(65) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3 = 1, R_2 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3 = 1:$$

$$\Rightarrow h_3 = 1 \text{ that is a contradiction.}$$

$$(66) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3 = 1, R_2 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2) \text{ is solvable, a contradiction.}$$

$$(69) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(70) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(71) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}(h_2^{-1}h_3)^2 = 1:$$

$$\Rightarrow h_3 = 1 \text{ that is a contradiction.}$$

$$(72) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-2}h_2^{-1}h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(73) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-3}h_2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(74) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-2}h_2h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(75) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}(h_3^{-1}h_2)^2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2^{-1}h_3 \rangle \text{ is abelian, a contradiction.}$$

$$(76) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^2h_3 = 1:$$

$$\Rightarrow h_2 = 1 \text{ that is a contradiction.}$$

$$(77) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2h_3^2 = 1:$$

By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.

$$(78) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(79) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2(h_2h_3^{-1})^2h_2^{-1}h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(80) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2(h_2h_3^{-1})^2h_3^{-1}h_2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle \text{ is abelian, a contradiction.}$$

$$(81) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2(h_2h_3^{-1})^2h_2h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(84) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2h_3h_2h_3^{-2}h_2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(85) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2h_3h_2h_3^{-1}h_2h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(86) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2h_3(h_2h_3^{-1})^2h_2 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(87) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2h_3^2h_2h_3^{-1}h_2 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(88) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2h_3^2h_2^{-2}h_3 = 1:$$

$$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle \text{ is abelian, a contradiction.}$$

$$(89) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(90) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2$ and $h_2 \mapsto h_2$, we have: $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2 h_3$. So $h_2^{-1} h_3^{-1} = (x^{h_2})^{-1}$. Also $(h_2 h_3)^2 = (h_2^{-1} h_3^{-1})^2$ so $H = \langle x, (x^{h_2})^{-1} \rangle = \langle x, x^{h_2} \rangle \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(91) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(92) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-2} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(93) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3^{-1} h_2 \rangle$ is abelian, a contradiction.

$$(94) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(95) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(96) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(97) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(99) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 h_3^{-1} \rangle$ is abelian, a contradiction.

$$(100) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(103) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(105) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have: $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(106) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(110) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(111) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(113) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(114) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(119) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(126) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

$$(127) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.

$$(128) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow h_3^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_2^{-1} h_3^{-1}$. So $h_3^{-1} h_2^{-1} = x^{h_3}$. Also $(h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_3} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_3 \rangle$. Since $\frac{G}{H} = \frac{\langle h_3 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(131) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-3} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(132) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(133) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(134) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(136) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(139) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(143) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (89) and with the same discussion, there is a contradiction.

$$(144) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(148) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(149) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (106) and with the same discussion, there is a contradiction.

$$(151) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-3} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(152) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(153) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (72) and with the same discussion, there is a contradiction.

$$(156) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2^{-1} h_3 \rangle$ is abelian, a contradiction.

$$(157) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(158) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (73) and with the same discussion, there is a contradiction.

$$(159) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (131) and with the same discussion, there is a contradiction.

$$(163) \quad R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(164) \quad R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (88) and with the same discussion, there is a contradiction.

$$(166) \quad R_1 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 h_2^{-1} \rangle$ is abelian, a contradiction.

$$(168) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^3 h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(169) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(171) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(172) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (90) and with the same discussion, there is a contradiction.

$$(174) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(175) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (133) and with the same discussion, there is a contradiction.

$$(176) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (44) and with the same discussion, there is a contradiction.

$$(178) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(180) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(181) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

Using Tietze transformation where $h_3 \mapsto h_2 h_3$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. So with the same discussion such as item (172), there is a contradiction.

$$(183) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (61) and with the same discussion, there is a contradiction.

$$(184) \quad R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

- (185) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (187) $R_1 : h_2^2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$:
 By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.
- (188) $R_1 : h_2^2 h_3^{-1} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.
- (189) $R_1 : h_2^2 h_3^{-1} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.
- (193) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (196) $R_1 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2^{-1} h_3 \rangle$ is abelian, a contradiction.
- (198) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (91) and with the same discussion, there is a contradiction.
- (199) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (200) $R_1 : h_2 (h_2 h_3^{-1})^3 h_2 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (201) $R_1 : h_2 (h_2 h_3^{-1})^3 h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (128) and with the same discussion, there is a contradiction.
- (202) $R_1 : h_2 (h_2 h_3^{-1})^3 h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3^{-1} h_2 \rangle$ is abelian, a contradiction.
- (203) $R_1 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (85) and with the same discussion, there is a contradiction.
- (205) $R_1 : h_2 h_3 h_2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (92) and with the same discussion, there is a contradiction.
- (207) $R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1$:
 By interchanging h_2 and h_3 in (55) and with the same discussion, there is a contradiction.
- (208) $R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (169) and with the same discussion, there is a contradiction.
- (210) $R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1$:
 By interchanging h_2 and h_3 in (200) and with the same discussion, there is a contradiction.
- (213) $R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (214) $R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (216) $R_1 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
 By interchanging h_2 and h_3 in (54) and with the same discussion, there is a contradiction.
- (217) $R_1 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 By interchanging h_2 and h_3 in (57) and with the same discussion, there is a contradiction.

$$(219) \quad R_1 : h_2 h_3^2 h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (163) and with the same discussion, there is a contradiction.

$$(220) \quad R_1 : h_2 h_3^2 h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (74) and with the same discussion, there is a contradiction.

$$(222) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (70) and with the same discussion, there is a contradiction.

$$(223) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(226) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (180) and with the same discussion, there is a contradiction.

$$(227) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (185) and with the same discussion, there is a contradiction.

$$(229) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (76) and with the same discussion, there is a contradiction.

$$(230) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(231) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (132) and with the same discussion, there is a contradiction.

$$(232) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (43) and with the same discussion, there is a contradiction.

$$(233) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(234) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (184) and with the same discussion, there is a contradiction.

$$(235) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (181) and with the same discussion, there is a contradiction.

$$(238) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (171) and with the same discussion, there is a contradiction.

$$(239) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (168) and with the same discussion, there is a contradiction.

$$(241) \quad R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (199) and with the same discussion, there is a contradiction.

$$(242) \quad R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (81) and with the same discussion, there is a contradiction.

$$(244) \quad R_1 : h_2 h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (84) and with the same discussion, there is a contradiction.

$$(247) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3^{-1} h_2 \rangle$ is abelian, a contradiction.

$$(249) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(250) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (94) and with the same discussion, there is a contradiction.

$$(251) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (134) and with the same discussion, there is a contradiction.

$$(254) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (95) and with the same discussion, there is a contradiction.

$$(259) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(264) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (213) and with the same discussion, there is a contradiction.

$$(265) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (113) and with the same discussion, there is a contradiction.

$$(266) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(270) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(272) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(273) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(275) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (86) and with the same discussion, there is a contradiction.

$$(276) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (214) and with the same discussion, there is a contradiction.

$$(278) \quad R_1 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (126) and with the same discussion, there is a contradiction.

$$(281) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (111) and with the same discussion, there is a contradiction.

$$(283) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (96) and with the same discussion, there is a contradiction.

$$(284) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(285) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (259) and with the same discussion, there is a contradiction.

$$(287) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (71) and with the same discussion, there is a contradiction.

$$(288) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(289) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (69) and with the same discussion, there is a contradiction.

$$(290) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (75) and with the same discussion, there is a contradiction.

$$(291) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (41) and with the same discussion, there is a contradiction.

$$(292) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (103) and with the same discussion, there is a contradiction.

$$(293) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (93) and with the same discussion, there is a contradiction.

$$(295) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(296) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (78) and with the same discussion, there is a contradiction.

$$(297) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (97) and with the same discussion, there is a contradiction.

$$(299) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (100) and with the same discussion, there is a contradiction.

$$(300) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (119) and with the same discussion, there is a contradiction.

$$(301) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (79) and with the same discussion, there is a contradiction.

$$(302) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (80) and with the same discussion, there is a contradiction.

$$(303) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (99) and with the same discussion, there is a contradiction.

$$(305) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (58) and with the same discussion, there is a contradiction.

$$(308) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(311) \quad R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.

$$(312) \quad R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (148) and with the same discussion, there is a contradiction.

$$(313) \quad R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (59) and with the same discussion, there is a contradiction.

$$(314) \quad R_1 : h_2 h_3^{-4} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(315) \quad R_1 : h_2 h_3^{-4} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(316) \quad R_1 : h_2 h_3^{-4} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(318) \quad R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(320) \quad R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (139) and with the same discussion, there is a contradiction.

$$(322) \quad R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (136) and with the same discussion, there is a contradiction.

$$(323) \quad R_1 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (166) and with the same discussion, there is a contradiction.

$$(325) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(326) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(328) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(329) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (40) and with the same discussion, there is a contradiction.

$$(331) \quad R_1 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (270) and with the same discussion, there is a contradiction.

$$(333) \quad R_1 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (193) and with the same discussion, there is a contradiction.

$$(335) \quad R_1 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (247) and with the same discussion, there is a contradiction.

$$(336) \quad R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (266) and with the same discussion, there is a contradiction.

$$(337) \quad R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(338) \quad R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(340) \quad R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(345) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (273) and with the same discussion, there is a contradiction.

$$(346) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (62) and with the same discussion, there is a contradiction.

$$(347) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (272) and with the same discussion, there is a contradiction.

$$(348) \quad R_1 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (196) and with the same discussion, there is a contradiction.

$$(349) \quad R_1 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (66) and with the same discussion, there is a contradiction.

$$(351) \quad R_1 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1:$$

By interchanging h_2 and h_3 in (202) and with the same discussion, there is a contradiction.

$$(355) \quad R_1 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (65) and with the same discussion, there is a contradiction.

8.17. $\mathbf{C}_5 - \mathbf{C}_5(- - \mathbf{C}_6 - -)$. By considering the relations from Tables 5 and 21 which are not disproved, it can be seen that there are 440 cases for the relations of two cycles C_5 and a cycle C_6 in the graph $C_5 - C_5(- - C_6 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 404 cases of these 440 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 36 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_6 - -)$ in $K(\alpha, \beta)$. These cases are listed in table 22. In the following, we show that all of these 36 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_6 - -)$.

$$(1) \quad R_1 : h_2^3 h_3^{-2} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2^2 h_3^4 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(2) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(3) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(4) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-4} h_2 = 1, \quad R_3 : h_2^4 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(5) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-3} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(6) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(7) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(8) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \quad R_3 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(9) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(10) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(11) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(12) \quad R_1 : h_2^2 h_3^{-3} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(13) \quad R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-2} h_2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

TABLE 22. The relations of a $C_5 - C_5(- - C_6 - -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3
1	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^4 = 1$
2	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1$
3	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$
4	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2^4 h_3^2 = 1$
5	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$	$h_2 h_3 h_2 h_3^{-3} h_2 = 1$
6	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
7	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1$
8	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$
9	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
10	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
11	$h_2^2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$
12	$h_2^2 h_3^{-3} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$
13	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_3 = 1$
14	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$
15	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$
16	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1$
17	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
18	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1$
19	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^2 h_3 = 1$
20	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$
21	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1$
22	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
23	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
24	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 (h_2 h_3^{-2})^2 h_2 = 1$
25	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$
26	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
27	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
28	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
29	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 (h_3 h_2^{-1} h_3)^2 = 1$
30	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
31	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 = 1$
32	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-2} h_3 = 1$
33	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
34	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
35	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2^2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
36	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3^2 = 1$

(14) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1$, $R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow h_2 = 1$ that is a contradiction.

- (15) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (16) $R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (17) $R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (18) $R_1 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^3 = 1:$
 By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.
- (19) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.
- (20) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1:$
 By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.
- (21) $R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1:$
 By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.
- (22) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (23) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.
- (24) $R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (25) $R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.
- (26) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (27) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (28) $R_1 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.
- (29) $R_1 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (30) $R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.
- (31) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (32) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1:$
 By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (33) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.
- (34) $R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(35) \ R_1 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, \ R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, \ R_3 : h_2^2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(36) \ R_1 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, \ R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, \ R_3 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$C_5 - C_5(-C_6 - -)$ subgraph: By considering the relations from Tables 5 and 21 which are not disproved, it can be seen that there are 2121 cases for the relations of two cycles C_5 and a cycle C_6 in the graph $C_5 - C_5(-C_6 - -)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 1907 cases of these 2121 cases are finite and solvable, that is a contradiction with the assumptions. So there are 214 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)$ in the graph $K(\alpha, \beta)$. These cases are listed in table 23. In the following, we show that 208 cases of these relations lead to a contradiction and just 6 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)$ in the graph $K(\alpha, \beta)$. Cases which are not disproved are marked by *s in the Table 23.

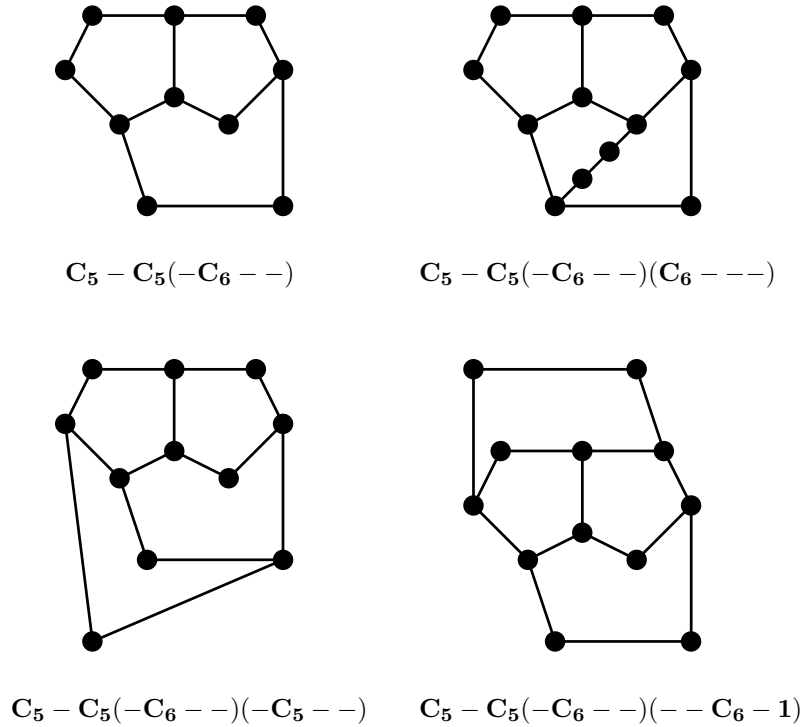


FIGURE 19. The graph $C_5 - C_5(-C_6 - -)$ and some forbidden subgraphs which contain it

$$(1) \ R_1 : h_2^3 h_3^2 = 1, \ R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \ R_3 : h_2^2 h_3^{-4} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

Table 23: The relations of a $C_5 - C_5(-C_6 - -)$

n	R_1	R_2	R_3
1	$h_2^3 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-4} h_2 = 1$
2	$h_2^3 h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^3 h_2 h_3^{-1} h_2 = 1$
3	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1$
4	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^3 = 1$
5	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
6	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$
7	$h_2^3 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 h_2^{-1} h_3 = 1$
8	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^4 = 1$
9	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$
10	$h_2^3 h_3^{-2} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1$
11	$h_2^3 h_3^{-2} h_2 = 1$	$(h_2 h_3^{-1})^3 h_2 h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
12	$h_2^2 h_3^3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^3 h_3^{-3} h_2 = 1$
13	$h_2^2 h_3^3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^3 h_3 h_2^{-1} h_3^2 = 1$
14*	$h_2^2 h_3^2 h_2^{-1} h_3 = 1$	$h_2^2 h_3^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
15	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1$
16	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$
17	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1$
18	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^3 h_2^{-1} h_3^2 = 1$
19	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^3 = 1$
20	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
21	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-3} h_2^{-1} h_3^3 = 1$
22	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^3 h_2^{-1} h_3^2 = 1$
23	$h_2^2 h_3 h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^3 = 1$
24	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
25	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
26	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
27	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2^{-1} h_3^3 = 1$
28	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^2 = 1$
29	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-2} h_3 = 1$
30	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
31	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1$
32	$h_2^2 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$h_3^2 (h_3 h_2^{-1})^3 h_3 = 1$
33	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^2 h_2 h_3 = 1$
34	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3 h_2^{-1} h_3 h_2 h_3 = 1$
35	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2 h_3 = 1$
36	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} (h_2 h_3)^2 = 1$
37	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_3 h_2^{-1} h_3 = 1$
38	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-2} h_3 = 1$
39	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3^2 = 1$

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Table 23 – continued from previous page

n	R_1	R_2	R_3
40	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1$
41	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2 h_3^{-2} h_2 = 1$
42	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1$
43	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
44	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 h_3 = 1$
45	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2^{-1} h_3^2 = 1$
46	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3^2 = 1$
47	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^3 = 1$
48	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1$
49	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$
50	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1$
51	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$
52	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 (h_2 h_3^{-1})^3 h_2 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
53	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3)^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^3 = 1$
54	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$
55	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
56	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2^4 h_3^2 = 1$
57	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1$
58	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
59	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$
60	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 (h_3 h_2^{-2})^2 h_3 = 1$
61	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
62	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1$
63	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$
64	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
65	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3^{-3} h_2^{-1} h_3^2 = 1$
66	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$
67	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$
68	$h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3^2 = 1$
69*	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-4} h_2 h_3 = 1$
70	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1$
71	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1$
72	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$
73	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$
74	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
75	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
76	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1$
77	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$
78	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$
79	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 (h_3 h_2^{-2})^2 h_3 = 1$

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Table 23 – continued from previous page

n	R_1	R_2	R_3
80	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$h_2(h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
81	$h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1$
82	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
83	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1$
84	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1$
85	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^3 = 1$
86	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 = 1$
87	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1$
88	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
89	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-3} h_2^{-1} h_3^2 = 1$
90	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^2 = 1$
91	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$
92	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2^2 (h_3 h_2^{-1} h_3)^2 = 1$
93	$h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2^2 (h_3 h_2^{-1})^2 h_3^2 = 1$
94	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1$
95	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
96	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
97	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1$
98	$h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$
99*	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^3 (h_2 h_3^{-1})^2 h_2 = 1$
100	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
101	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^2 h_2^{-2} h_3 = 1$
102	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
103	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$
104	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1$
105	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$
106	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
107	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
108	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
109	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$
110	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
111	$h_2^2 h_3^{-1} h_2^2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3^2 = 1$
112*	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2^2 h_3^{-1} h_2 h_3^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
113	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1$
114	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2^2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1$
115	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
116	$h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
117	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
118	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
119	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-2} h_2 h_3 = 1$

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Table 23 – continued from previous page

n	R_1	R_2	R_3
120*	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$
121	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
122	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1$
123	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$
124	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
125	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2^3 h_3^3 = 1$
126	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$
127	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_3 = 1$
128	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-2} h_2 h_3^2 = 1$
129	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1$
130	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$
131	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 (h_3 h_2^{-1} h_3)^2 = 1$
132	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
133	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
134	$h_2 h_3^2 h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$
135	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$
136	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1$
137	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-4} h_2 h_3 = 1$
138	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$
139	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 (h_2 h_3^{-2})^2 h_2 = 1$
140	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$
141	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1$
142	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
143	$h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
144	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^3 h_2^{-1} h_3^2 = 1$
145	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^3 = 1$
146	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$
147	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$(h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$
148	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^2 = 1$
149	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2^{-2} h_3 = 1$
150	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$
151	$h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1$
152	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$	$h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$
153	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2^2 h_3^4 = 1$
154	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-4} h_2 = 1$	$h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$
155	$h_2 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$
156	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 (h_2 h_3)^2 h_3 = 1$
157	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} (h_2 h_3)^2 = 1$
158	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$
159	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3)^2 h_3 h_2^{-1} h_3 = 1$

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Table 23 – continued from previous page

n	R_1	R_2	R_3
160	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$
161	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1$
162	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2 h_3^{-1} h_2 h_3 = 1$
163	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$
164	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3 h_2^{-2} h_3 = 1$
165	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^{-2} h_3 = 1$
166	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-3} h_3 = 1$
167	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
168	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^2 = 1$
169	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1$
170	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1$
171	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1$
172	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1$
173	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
174	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-2} h_3 = 1$
175	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
176	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$
177	$h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$	$h_3 (h_3 h_2^{-1})^3 h_3 = 1$	$h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1$
178	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1$
179	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
180	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2^3 h_3^{-1} h_2^{-2} h_3 = 1$
181	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
182*	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$
183	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$
184	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$
185	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$	$h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1$
186	$h_2 h_3^{-3} h_2 h_3 = 1$	$h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$	$h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$
187	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1$
188	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1$
189	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1$
190	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^2 h_3 = 1$
191	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$
192	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^3 h_3^{-1} h_2^2 h_3 = 1$
193	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$
194	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^3 h_3^{-1} h_2^{-2} h_3 = 1$
195	$h_2 h_3^{-2} h_2 h_3^2 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$	$h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$
196	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3^2 = 1$
197	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$(h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$
198	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$h_2^3 h_3^{-1} h_2^2 h_3 = 1$
199	$h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$	$(h_2 h_3^{-1})^2 h_2 h_3^2 = 1$	$h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$

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Table 23 – continued from previous page

n	R_1	R_2	R_3
200	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3 = 1$	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3 = 1$	$h_2^3h_3^{-1}h_2^{-1}h_3^2 = 1$
201	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3 = 1$	$h_2h_3^{-1}(h_3^{-1}h_2)^2h_3 = 1$	$h_2^3h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1$
202	$h_2h_3^{-1}h_2h_3^3 = 1$	$(h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$	$h_2^2h_3^{-1}h_2h_3h_2^{-2}h_3 = 1$
203	$h_2h_3^{-1}h_2h_3^3 = 1$	$(h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$	$h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3^2 = 1$
204	$h_2h_3^{-1}h_2h_3^3 = 1$	$(h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-2}h_3^2 = 1$
205	$h_2h_3^{-1}h_2h_3^2h_2^{-1}h_3 = 1$	$h_2h_3^{-1}h_2h_3^2h_2^{-1}h_3 = 1$	$h_2^2h_3^{-1}h_2(h_3h_2^{-1})^2h_3 = 1$
206	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$h_2^3h_3^{-1}h_2^2h_3 = 1$
207	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$h_2^2h_3h_2^2h_3^{-1}h_2 = 1$
208	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$h_2h_3^{-2}h_2^{-1}h_3h_2^{-2}h_3 = 1$
209	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$(h_2h_3^{-1})^2h_2^{-1}h_3^2 = 1$	$h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1$
210	$(h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$	$h_3^2(h_3h_2^{-1})^2h_3 = 1$	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$
211	$(h_2h_3^{-1})^2h_3^{-1}h_2^{-1}h_3 = 1$	$(h_2h_3^{-1})^2h_3^{-1}h_2^{-1}h_3 = 1$	$h_2^2h_3^{-1}h_2h_3^2h_2^{-1}h_3 = 1$
212	$(h_2h_3^{-1})^2h_3^{-1}h_2^{-1}h_3 = 1$	$(h_2h_3^{-1})^2h_3^{-1}h_2^{-1}h_3 = 1$	$h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$
213	$h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$	$(h_2h_3^{-1})^3h_2h_3 = 1$	$h_2h_3^{-1}h_2^{-1}h_3h_2^{-2}h_3^2 = 1$
214	$h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$	$(h_2h_3^{-1})^3h_2h_3 = 1$	$h_2h_3^{-3}h_2^2h_3^{-1}h_2 = 1$

- (2) $R_1 : h_2^3h_3^2 = 1$, $R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$, $R_3 : h_2h_3^3h_2h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (3) $R_1 : h_2^3h_3h_2^{-1}h_3 = 1$, $R_2 : h_2h_3^{-1}h_2h_3^3 = 1$, $R_3 : h_2h_3h_2^{-1}h_3h_2h_3^{-1}h_2h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (4) $R_1 : h_2^3h_3h_2^{-1}h_3 = 1$, $R_2 : h_2h_3^{-1}h_2h_3^3 = 1$, $R_3 : (h_2h_3^{-1})^2h_2^{-1}h_3^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^3h_3h_2^{-1}h_3 = 1$, $R_2 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^2h_2^{-1}h_3h_2h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $R_1 : h_2^3h_3h_2^{-1}h_3 = 1$, $R_2 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-2}h_2 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (7) $R_1 : h_2^3h_3h_2^{-1}h_3 = 1$, $R_2 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^{-2}h_2h_3^2h_2^{-1}h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (8) $R_1 : h_2^3h_3^{-2}h_2 = 1$, $R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$, $R_3 : h_2^2h_3^4 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (9) $R_1 : h_2^3h_3^{-2}h_2 = 1$, $R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (10) $R_1 : h_2^3h_3^{-2}h_2 = 1$, $R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$, $R_3 : h_2h_3h_2^{-1}(h_2^{-1}h_3)^2h_3 = 1$:
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (11) $R_1 : h_2^3h_3^{-2}h_2 = 1$, $R_2 : (h_2h_3^{-1})^3h_2h_3 = 1$, $R_3 : h_2^2h_3^{-2}(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (12) $R_1 : h_2^2h_3^3 = 1$, $R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1$, $R_3 : h_2^3h_3^{-3}h_2 = 1$:

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(13) R_1 : h_2^2 h_3^3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2^3 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(15) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(16) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(17) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(18) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.

$$(19) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.

$$(20) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 5)$ is solvable, a contradiction.

$$(21) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3^{-3} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.

$$(22) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.

$$(23) R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.

$$(24) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(25) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(26) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(27) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-3} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.

$$(28) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(29) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(30) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$R_2 \Rightarrow R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$. So, $R_3 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(31) R_1 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

- (32) $R_1 : h_2^2(h_3h_2^{-1})^2h_3 = 1, R_2 : (h_2h_3^{-1})^2h_2h_3^2 = 1, R_3 : h_3^2(h_3h_2^{-1})^3h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (33) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2h_3^2h_2h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.
- (34) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2h_3h_2^{-1}h_3h_2h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2^{-1}h_3^{-1} \rangle$ is abelian, a contradiction.
- (35) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2h_3^{-1}h_2^{-1}h_3h_2h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (36) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2h_3^{-1}(h_2h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (37) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : (h_2h_3)^2h_3h_2^{-1}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (38) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (39) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (40) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : (h_2h_3)^2(h_2^{-1}h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (41) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^2h_2h_3^{-2}h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (42) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^{-2}h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (43) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (44) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-2}(h_2^{-1}h_3)^2h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (45) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-3}h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (46) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-3}h_2h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (47) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-2}h_2h_3^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (48) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-2}h_2h_3h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (49) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-2}h_2h_3^{-1}h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (50) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2h_3^{-1}(h_3^{-1}h_2)^2h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (51) $R_1 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1, R_2 : h_2(h_2h_3^{-1})^3h_2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-1}(h_3^{-1}h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

- (52) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^3 h_2 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (53) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^3 = 1:$
 $R_2 \Rightarrow R_3 : h_3^2 h_2^{-1} h_3^{-2} h_2 = 1$ and $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$. Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_3 : (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$. So, $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (54) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (55) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (56) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1, R_3 : h_2^4 h_3^2 = 1:$
 By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.
- (57) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (58) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-4} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 $R_1 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3^{-1} h_2^{-1}$. So $h_2^{-1} h_3^{-1} = x^{h_2}$.
 R_1 and $R_3 \Rightarrow (h_3^{-1} h_2^{-1})^2 = (h_2^{-1} h_3^{-1})^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (59) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (60) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (61) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 With the same discussion such as item (58), there is a contradiction.
- (62) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (63) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (64) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$
 With the same discussion such as item (43), there is a contradiction.
- (65) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3^{-3} h_2^{-1} h_3^2 = 1:$
 With the same discussion such as item (45), there is a contradiction.
- (66) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (67) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$
 With the same discussion such as item (66), there is a contradiction.

- (68) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (70) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (71) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (72) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (73) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (74) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (75) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (76) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (77) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (78) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (79) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (80) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (81) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (82) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (83) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (84) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (85) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (86) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (87) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (88) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (89) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-3} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -4)$ is solvable, a contradiction.
- (90) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (91) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (92) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.
- (93) $R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 (h_3 h_2^{-1})^2 h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (94) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3)$ is solvable, a contradiction.
- (95) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (96) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 By interchanging h_2 and h_3 in (95) and with the same discussion, there is a contradiction.
- (97) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (98) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (100) $R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (101) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_3 : h_2^2 h_3^2 h_2^{-2} h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (102) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_3 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (103) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (104) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (105) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_3^3 (h_3 h_2^{-1})^2 h_3 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (106) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (107) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1:$
 $\Rightarrow h_2 = 1$ that is a contradiction.
- (108) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (109) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.

- (110) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (111) $R_1 : h_2^2 h_3^{-1} h_2^2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (113) $R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (114) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow h_3 = 1$ that is a contradiction.
- (115) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (116) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$
 $R_2 \Rightarrow R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$. So, $R_3 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.
- (117) $R_1 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$
 With the same discussion such as item (116), there is a contradiction.
- (118) $R_1 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (119) $R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^{-2} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (121) $R_1 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$
 By interchanging h_2 and h_3 in (100) and with the same discussion, there is a contradiction.
- (122) $R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1:$
 By interchanging h_2 and h_3 in (84) and with the same discussion, there is a contradiction.
- (123) $R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$
 By interchanging h_2 and h_3 in (83) and with the same discussion, there is a contradiction.
- (124) $R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 By interchanging h_2 and h_3 in (82) and with the same discussion, there is a contradiction.
- (125) $R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^3 h_3^3 = 1:$
 By interchanging h_2 and h_3 in (85) and with the same discussion, there is a contradiction.
- (126) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$
 By interchanging h_2 and h_3 in (103) and with the same discussion, there is a contradiction.
- (127) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (104) and with the same discussion, there is a contradiction.
- (128) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3^2 = 1:$
 By interchanging h_2 and h_3 in (101) and with the same discussion, there is a contradiction.
- (129) $R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1:$
 By interchanging h_2 and h_3 in (102) and with the same discussion, there is a contradiction.

$$(130) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (107) and with the same discussion, there is a contradiction.

$$(131) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (111) and with the same discussion, there is a contradiction.

$$(132) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (110) and with the same discussion, there is a contradiction.

$$(133) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (109) and with the same discussion, there is a contradiction.

$$(134) R_1 : h_2 h_3^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (108) and with the same discussion, there is a contradiction.

$$(135) R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (118) and with the same discussion, there is a contradiction.

$$(136) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(137) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-4} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(138) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3)$ is solvable, a contradiction.

$$(139) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(140) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (139) and with the same discussion, there is a contradiction.

$$(141) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(142) R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(143) R_1 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$R_2 \Rightarrow R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. Using Tietze transformation where $h_3 \mapsto h_3 h_2^{-1}$ and $h_2 \mapsto h_2$, we have $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : (h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$. So, $R_3 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(144) R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 4)$ is solvable, a contradiction.

$$(145) R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.

$$(146) R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(147) R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(148) \quad R_1 : h_2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^2h_3h_2^{-2}h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(149) \quad R_1 : h_2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(150) \quad R_1 : h_2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(151) \quad R_1 : h_2(h_3h_2^{-1})^2h_3^{-1}h_2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-2}h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(152) \quad R_1 : h_2(h_3h_2^{-1})^3h_3 = 1, R_2 : h_2h_3^{-1}(h_2^{-1}h_3)^2h_3 = 1, R_3 : h_2h_3h_2^{-1}h_3^2h_2^{-2}h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(153) \quad R_1 : h_2(h_3h_2^{-1})^3h_3 = 1, R_2 : h_2h_3^{-4}h_2 = 1, R_3 : h_2^2h_3^4 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(154) \quad R_1 : h_2(h_3h_2^{-1})^3h_3 = 1, R_2 : h_2h_3^{-4}h_2 = 1, R_3 : h_2^2h_3^{-3}h_2^{-1}h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(155) \quad R_1 : h_2(h_3h_2^{-1})^3h_3 = 1, R_2 : h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1, R_3 : h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3^2 = 1:$$

$\Rightarrow h_3 = 1$ that is a contradiction.

$$(156) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2(h_2h_3)^2h_3 = 1:$$

By interchanging h_2 and h_3 in (33) and with the same discussion, there is a contradiction.

$$(157) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^2h_3^{-1}(h_2h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.

$$(158) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1:$$

By interchanging h_2 and h_3 in (39) and with the same discussion, there is a contradiction.

$$(159) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : (h_2h_3)^2h_3h_2^{-1}h_3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(160) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3h_2h_3^{-2}h_2h_3 = 1:$$

By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.

$$(161) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3(h_2h_3^{-1})^2h_2h_3 = 1:$$

By interchanging h_2 and h_3 in (40) and with the same discussion, there is a contradiction.

$$(162) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3^2h_2h_3^{-1}h_2h_3 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(163) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^{-1}h_2h_3 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(164) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^3h_3h_2^{-2}h_3 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(165) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^3h_3^{-1}h_2^{-2}h_3 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(166) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^2h_3h_2^{-3}h_3 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(167) \quad R_1 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1, R_3 : h_2^2h_3h_2^{-2}h_3^{-1}h_2 = 1:$$

By interchanging h_2 and h_3 in (43) and with the same discussion, there is a contradiction.

$$(168) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (41) and with the same discussion, there is a contradiction.

$$(169) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (50) and with the same discussion, there is a contradiction.

$$(170) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.

$$(171) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (44) and with the same discussion, there is a contradiction.

$$(172) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (48) and with the same discussion, there is a contradiction.

$$(173) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (49) and with the same discussion, there is a contradiction.

$$(174) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (65) and with the same discussion, there is a contradiction.

$$(175) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (64) and with the same discussion, there is a contradiction.

$$(176) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(177) R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_2 : h_3 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (51) and with the same discussion, there is a contradiction.

$$(178) R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (86) and with the same discussion, there is a contradiction.

$$(179) R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (87) and with the same discussion, there is a contradiction.

$$(180) R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (89) and with the same discussion, there is a contradiction.

$$(181) R_1 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (88) and with the same discussion, there is a contradiction.

$$(183) R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (71) and with the same discussion, there is a contradiction.

$$(184) R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (70) and with the same discussion, there is a contradiction.

$$(185) R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (81) and with the same discussion, there is a contradiction.

$$(186) R_1 : h_2 h_3^{-3} h_2 h_3 = 1, R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (80) and with the same discussion, there is a contradiction.

$$(187) R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_3 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(188) R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(189) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(190) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(191) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(192) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(193) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(194) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(195) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(196) \quad R_1 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(197) \quad R_1 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : (h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(198) \quad R_1 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(199) \quad R_1 : h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(200) \quad R_1 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (136) and with the same discussion, there is a contradiction.

$$(201) \quad R_1 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (137) and with the same discussion, there is a contradiction.

$$(202) \quad R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(203) \quad R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(204) \quad R_1 : h_2 h_3^{-1} h_2 h_3^3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(205) \quad R_1 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (113) and with the same discussion, there is a contradiction.

$$(206) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (144) and with the same discussion, there is a contradiction.

$$(207) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (145) and with the same discussion, there is a contradiction.

$$(208) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (146) and with the same discussion, there is a contradiction.

$$(209) R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (147) and with the same discussion, there is a contradiction.

$$(210) R_1 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

$R_1 \Rightarrow R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$ and $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$. So, $R_3 \Rightarrow h_2^2 \in Z(G)$ where $G = \langle h_2, h_3 \rangle$. Let $x = h_3 h_2^{-1}$. So $h_2^{-1} h_3 = x^{h_2}$. Also $(h_3 h_2^{-1})^2 = (h_2^{-1} h_3)^2$ so if $H = \langle x, x^{h_2} \rangle \Rightarrow H \cong BS(1, -1)$ is solvable. By Corollary 8.2 $H \trianglelefteq G$ since $G = \langle x, h_2 \rangle$. Since $\frac{G}{H} = \frac{\langle h_2 \rangle H}{H}$ is a cyclic group, it is solvable. So G is solvable, a contradiction.

$$(211) R_1 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (114) and with the same discussion, there is a contradiction.

$$(212) R_1 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (115) and with the same discussion, there is a contradiction.

$$(213) R_1 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

$$(214) R_1 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow h_2 = 1$ that is a contradiction.

8.18. **$C_5 - C_5(-C_6 - -)(C_6 - - -)$** . By considering the relations from Tables 5 and 23 which are not disproved, it can be seen that there are 56 cases for the relations of two cycles C_5 and two cycles C_6 in the graph $C_5 - C_5(-C_6 - -)(C_6 - - -)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 54 cases of these 56 cases are finite and solvable, that is a contradiction with the assumptions. So there are just 2 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(C_6 - - -)$ in the graph $K(\alpha, \beta)$. These cases are listed in table 24. In the following, we show that these 2 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(C_6 - - -)$.

$$(1) R_1 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2 h_3 = 1, R_3 : h_2^3 (h_2 h_3^{-1})^2 h_2 = 1, R_4 : (h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(2) R_1 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-2} h_3 = 1, R_3 : h_3^3 (h_3 h_2^{-1})^2 h_3 = 1, R_4 : (h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

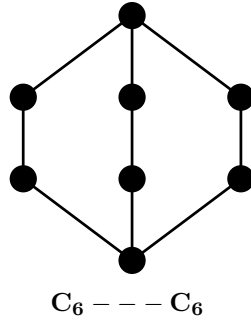
TABLE 24. The relations of a $C_4 - C_6(-C_6 - -)(C_6 - - -)$ in $K(\alpha, \beta)$

n	R_1	R_2	R_3	R_4
1	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^2 h_3^{-2} h_2 h_3 = 1$	$h_2^3 (h_2 h_3^{-1})^2 h_2 = 1$	$(h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1$
2	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_2 h_3^2 h_2^{-2} h_3 = 1$	$h_3^3 (h_3 h_2^{-1})^2 h_3 = 1$	$(h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1$

8.19. **$C_5 - C_5(-C_6 - -)(- - C_6 - 1)$** . By considering the relations from Tables 5 and 23 which are not disproved, it can be seen that there are 56 cases for the relations of two cycles C_5 and two cycles C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 1)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these

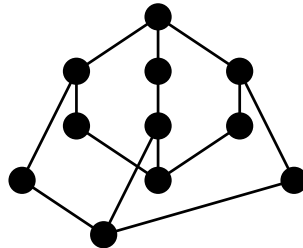
groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 1)$.

8.20. $C_5 - C_5(-C_6 - -)(-C_5 - -)$. By considering the relations from Tables 4 and 23 which are not disproved, it can be seen that there are 14 cases for the relations of three cycles C_5 and one cycle C_6 in the graph $C_5 - C_5(-C_6 - -)(-C_5 - -)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(-C_5 - -)$.



$C_6 - - - C_6$ subgraph: Suppose that there are two cycles of length 6 with three successive common edges in the graph $K(\alpha, \beta)$ and denote such subgraph by $C_6 - - - C_6$.

Let $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6]$ and $[a_1, b_1, a_2, b_2, a_3, b_3, a'_4, b'_4, a'_5, b'_5, a'_6, b'_6]$ be 12-tuples related to the cycles C_6 in the graph $C_6 - - - C_6$, where the first six components of these tuples are related to the three successive common edges of C_6 and C_6 . Without loss of generality we may assume that $a_1 = 1$, where $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6, a'_4, b'_4, a'_5, b'_5, a'_6, b'_6 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. With the same discussion such as about $K_{2,3}$, it is easy to see that $a_4 \neq a'_4$ and $b_6 \neq b'_6$. By considering the relations from Table 5 which are not disproved and above assumptions, it can be seen that there are 4454 cases for existing two cycles of length 6 with three successive common edges in the graph $K(\alpha, \beta)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and two relations which are between 4022 cases of these 4454 cases are solvable or finite. So there are 432 cases for the relations of the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 333 cases of these relations lead to contradictions and 99 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$.



21) $C_6 - - - C_6(C_6 - - - C_6)$

8.21. **$C_6 - - - C_6(C_6 - - - C_6)$** . By considering the 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$, it can be seen that there are 42 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 26 cases of these 42 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - - C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - - C_6)$.

- (1) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (3) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (4) $R_1 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (5) $R_1 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (6) $R_1 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (7) $R_1 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$, $R_3 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1$,
 $R_4 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (8) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-2})^2 h_2 h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (9) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$,
 $R_4 : (h_2 h_3^{-2})^2 h_2 h_3 = 1$:
By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.
- (10) $R_1 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.

$$(11) \quad R_1 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(12) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \\ R_4 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(13) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(14) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(15) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

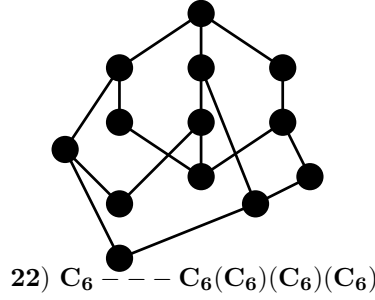
$$(16) \quad R_1 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$C_6 - - - C_6(C_6)$ subgraph: By considering the 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 2618 cases for the relations of three C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 2352 cases of these 2618 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 266 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 228 cases of these relations lead to contradictions and 38 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)$ in the graph $K(\alpha, \beta)$.

$C_6 - - - C_6(C_6)(C_6)$ subgraph: By considering the 38 cases related to the existence of $C_6 - - - C_6(C_6)$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 374 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 340 cases of these 374 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 34 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 30 cases of these relations lead to contradictions and 4 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)$ in the graph $K(\alpha, \beta)$.

8.22. **$C_6 - - - C_6(C_6)(C_6)(C_6)$.** By considering the 4 cases related to the existence of $C_6 - - - C_6(C_6)(C_6)$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 10 cases for the relations of five C_6 cycles in this structure. Using GAP [9], we see that all



groups with two generators h_2 and h_3 and five relations which are between 6 cases of these 10 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)(C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6)(C_6)(C_6)$.

$$(1) \quad R_1 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1, \quad R_2 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1, \quad R_5 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle \text{ is abelian, a contradiction.}$$

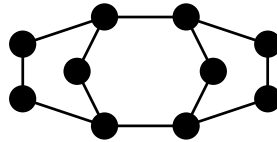
$$(2) \quad R_1 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, \quad R_2 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, \quad R_5 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

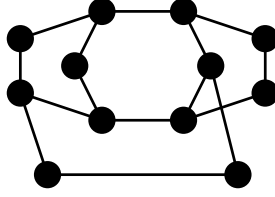
$$(3) \quad R_1 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1, \quad R_5 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2) \text{ is solvable, a contradiction.}$$

$$(4) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \quad R_5 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.



$C_5(- - C_6 - -)C_5$ subgraph: It can be seen that there are 2215 cases for the relations of two C_5 and one C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 2046 cases of these 2215 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 169 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 163 cases of these relations lead to contradictions and 6 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5$ in the graph $K(\alpha, \beta)$.



23) $C_5(- - C_6 - -)C_5(- - - C_6)$

8.23. $C_5(- - C_6 - -)C_5(- - - C_6)$. By considering the 6 cases related to the existence of $C_5(- - C_6 - -)C_5$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 134 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 130 cases of these 134 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(- - - C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(- - - C_6)$.

$$(1) \ R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \ R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \ R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(2) \ R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \ R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \ R_3 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

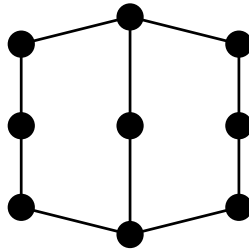
$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(3) \ R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \ R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \ R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(4) \ R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \ R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, \ R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1:$$

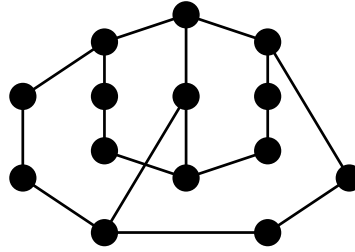
By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.



$C_6 - - C_6$

$C_6 - - C_6$ subgraph: Suppose that there are two cycles of length 6 with two successive common edges in the graph $K(\alpha, \beta)$ and denote such subgraph by $C_6 - - C_6$.

Let $[a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6]$ and $[a_1, b_1, a_2, b_2, a'_3, b'_3, a'_4, b'_4, a'_5, b'_5, a'_6, b'_6]$ be 12-tuples related to the cycles C_6 in the graph $C_6 - -C_6$, where the first four components of these tuples are related to the two successive common edges of C_6 and C_6 . Without loss of generality we may assume that $a_1 = 1$, where $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6, a'_3, b'_3, a'_4, b'_4, a'_5, b'_5, a'_6, b'_6 \in \text{supp}(\alpha)$ and $\alpha = 1 + h_2 + h_3$. With the same discussion such as about $K_{2,3}$, it is easy to see that $a_3 \neq a'_3$ and $b_6 \neq b'_6$. By considering the relations from Table 5 which are not disproved and above assumptions, it can be seen that there are 16462 cases for existing two cycles of length 6 with two successive common edges in the graph $K(\alpha, \beta)$. Using Gap [9], we see that all groups with two generators h_2 and h_3 and two relations which are between 14846 cases of these 16462 cases are solvable or finite. So there are 1616 cases for the relations of the existence of $C_6 - -C_6$ in the graph $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 620 cases of these relations lead to contradictions and 996 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6$ in the graph $K(\alpha, \beta)$.



24) $C_6 - -C_6(C_6 - -C_6)$

8.24. $C_6 - -C_6(C_6 - -C_6)$. By considering the 996 cases related to the existence of $C_6 - -C_6$ in the graph $K(\alpha, \beta)$, it can be seen that there are 5119 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 4983 cases of these 5119 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 136 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(C_6 - -C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 136 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - -C_6(C_6 - -C_6)$.

$$(1) \quad R_1 : h_2^3 h_3^3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(2) \quad R_1 : h_2^3 h_3^3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(3) \quad R_1 : h_2^3 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1) \text{ is solvable, a contradiction.}$$

$$(4) \quad R_1 : h_2^3 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1) \text{ is solvable, a contradiction.}$$

- (5) $R_1 : h_2^3 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2^2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^3 h_3^{-1} h_2^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (7) $R_1 : h_2^3 h_3^{-1} h_2^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (8) $R_1 : h_2^3 h_3^{-1} h_2 h_3^2 = 1$, $R_2 : h_2^2 h_3 h_2 h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2^3 h_3^{-1} h_2 h_3^2 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (10) $R_1 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3)$ is solvable, a contradiction.
- (11) $R_1 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3)$ is solvable, a contradiction.
- (12) $R_1 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (13) $R_1 : h_2^2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (14) $R_1 : h_2^2 h_3 (h_2 h_3^{-1})^2 h_2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (15) $R_1 : h_2^2 h_3^3 h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^2 h_2 h_3 h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3^3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.
- (16) $R_1 : h_2^2 h_3 (h_3 h_2^{-1})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (17) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^2 h_2^{-1} (h_3^{-1} h_2)^2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

- (18) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (19) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3)$ is solvable, a contradiction.
- (20) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (21) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (22) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (23) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2^2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (24) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 4)$ is solvable, a contradiction.
- (25) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^2 h_2 h_3^2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$:
By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.
- (26) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3)$ is solvable, a contradiction.
- (27) $R_1 : h_2^2 h_3 h_2^{-2} h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1$, $R_3 : h_2 (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -3)$ is solvable, a contradiction.
- (28) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (29) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (30) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(31) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(32) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^3 h_2^{-1} h_3^2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(33) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(34) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(35) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_2 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(36) \quad R_1 : h_2^2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \quad R_2 : (h_2^2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^2 h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(37) \quad R_1 : h_2^2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(38) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(39) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(40) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(41) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(42) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

$\Rightarrow h_3 = 1$, a contradiction.

$$(43) \quad R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(44) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, \\ R_4 : h_2h_3h_2h_3^{-1}h_2^{-2}h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(-1, 1) \text{ is solvable, a contradiction.}$$

$$(45) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, \\ R_4 : h_2h_3h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1: \\ \Rightarrow G \text{ has a torsion element, a contradiction.}$$

$$(46) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(47) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(48) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2(h_2^{-1}h_3)^2 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(49) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^3 = 1, \\ R_4 : h_2h_3^2h_2^{-1}h_3^3 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(50) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2h_3h_2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, \\ R_4 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 3) \text{ is solvable, a contradiction.}$$

$$(51) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2h_3h_2h_3^{-1}h_2^{-2}h_3 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, \\ R_4 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(52) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2h_3h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1, R_3 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, \\ R_4 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1: \\ \Rightarrow h_3 = 1, \text{ a contradiction.}$$

$$(53) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2h_3h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, \\ R_4 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2) \text{ is solvable, a contradiction.}$$

$$(54) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2h_3 = 1, R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow h_3 = 1, \text{ a contradiction.}$$

$$(55) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(56) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(57) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : (h_2h_3)^2(h_2^{-1}h_3)^2 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(58) \quad R_1 : h_2^2(h_3h_2^{-1})^2h_3^2 = 1, R_2 : (h_2h_3^{-1})^2h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^3 = 1, \\ R_4 : h_2h_3^2h_2^{-1}h_3^3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(59) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1) \text{ is solvable, a contradiction.}$$

$$(60) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1) \text{ is solvable, a contradiction.}$$

$$(61) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1, R_3 : (h_2h_3)^2(h_2^{-1}h_3)^2 = 1, \\ R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(62) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2(h_3h_2^{-1})^2h_3 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^3 = 1, \\ R_4 : h_2h_3^2h_2^{-1}h_3^3 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(63) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1, R_3 : (h_2h_3)^2h_2^{-2}h_3 = 1, \\ R_4 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1) \text{ is solvable, a contradiction.}$$

$$(64) \quad R_1 : h_2^2h_3^{-1}h_2^{-2}h_3^2 = 1, R_2 : h_2h_3^{-2}h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1, R_3 : (h_2h_3)^2(h_2^{-1}h_3)^2 = 1, \\ R_4 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-2}h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(65) \quad R_1 : h_2^2(h_3^{-1}h_2^{-1})^2h_3 = 1, R_2 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow G \text{ has a torsion element, a contradiction.}$$

$$(66) \quad R_1 : h_2^2(h_3^{-1}h_2^{-1})^2h_3 = 1, R_2 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2h_3 = 1, R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, \\ R_4 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2) \text{ is solvable, a contradiction.}$$

$$(67) \quad R_1 : h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2h_3 = 1, R_2 : h_2h_3(h_2h_3^{-1})^2h_3^{-1}h_2 = 1, R_3 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1, \\ R_4 : h_2h_3^{-2}h_2h_3h_2h_3^{-1}h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle \text{ is abelian, a contradiction.}$$

$$(68) \quad R_1 : h_2^2h_3^{-1}h_2^{-1}h_3h_2h_3 = 1, R_2 : h_2h_3^{-2}h_2h_3^3 = 1, R_3 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, \\ R_4 : (h_2h_3^{-1})^2h_2h_3^2h_2^{-1}h_3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(69) \quad R_1 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1, R_2 : h_2h_3h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1, R_3 : h_2h_3^{-3}h_2^2h_3^{-1}h_2 = 1, \\ R_4 : (h_2h_3^{-1}h_2)^2h_3^2 = 1:$$

By interchanging h_2 and h_3 in (39) and with the same discussion, there is a contradiction.

$$(70) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_3 : h_2 h_3^3 h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(71) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2) \text{ is solvable, a contradiction.}$$

$$(72) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(73) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1: \\ \Rightarrow G \text{ has a torsion element, a contradiction.}$$

$$(74) \quad R_1 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(75) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(76) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(77) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(78) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 h_3^3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(79) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.

$$(80) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(81) \quad R_1 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \quad R_2 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(82) \quad R_1 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \quad R_2 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(83) \quad R_1 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(84) \quad R_1 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(85) \quad R_1 : (h_2 h_3)^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 h_3 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(86) \quad R_1 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.

$$(87) \quad R_1 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(88) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (65) and with the same discussion, there is a contradiction.

$$(89) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (66) and with the same discussion, there is a contradiction.

$$(90) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(91) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (55) and with the same discussion, there is a contradiction.

$$(92) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (59) and with the same discussion, there is a contradiction.

$$(93) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.

$$(94) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (63) and with the same discussion, there is a contradiction.

$$(95) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (83) and with the same discussion, there is a contradiction.

$$(96) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, \quad R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(97) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, \quad R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(98) \quad R_1 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (60) and with the same discussion, there is a contradiction.

$$(99) \quad R_1 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (82) and with the same discussion, there is a contradiction.

$$(100) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (73) and with the same discussion, there is a contradiction.

$$(101) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(102) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(103) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.

$$(104) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.

$$(105) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (71) and with the same discussion, there is a contradiction.

$$(106) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(107) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$$(108) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (64) and with the same discussion, there is a contradiction.

$$(109) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (86) and with the same discussion, there is a contradiction.

$$(110) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (48) and with the same discussion, there is a contradiction.

$$(111) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (57) and with the same discussion, there is a contradiction.

$$(112) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (61) and with the same discussion, there is a contradiction.

$$(113) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_3^{-1} h_2)^2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (72) and with the same discussion, there is a contradiction.

$$(114) \quad R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (81) and with the same discussion, there is a contradiction.

$$(115) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (44) and with the same discussion, there is a contradiction.

$$(116) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (50) and with the same discussion, there is a contradiction.

$$(117) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (51) and with the same discussion, there is a contradiction.

$$(118) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (54) and with the same discussion, there is a contradiction.

$$(119) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(120) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(121) \quad R_1 : h_2 h_3^2 (h_2 h_3^{-1})^2 h_2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (53) and with the same discussion, there is a contradiction.

$$(122) \quad R_1 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(123) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (33) and with the same discussion, there is a contradiction.

$$(124) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (67) and with the same discussion, there is a contradiction.

$$(125) \quad R_1 : h_2 h_3^2 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 (h_2 h_3^{-1})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (30) and with the same discussion, there is a contradiction.

$$(126) \quad R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(127) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (87) and with the same discussion, there is a contradiction.

$$(128) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (106) and with the same discussion, there is a contradiction.

$$(129) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-3} h_3^2 h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.

$$(130) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (101) and with the same discussion, there is a contradiction.

$$(131) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(132) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (102) and with the same discussion, there is a contradiction.

$$(133) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1} h_3)^2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (104) and with the same discussion, there is a contradiction.

$$(134) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3^2 = 1, \\ R_4 : (h_2 h_3^{-2})^2 h_2^{-1} h_3 = 1:$$

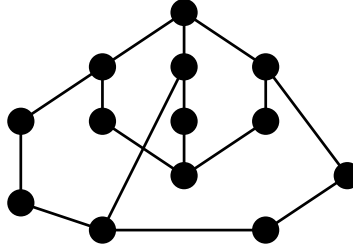
By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(135) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(136) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^3 = 1, \quad R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1} h_3)^2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.



25) $C_6 - - - C_6(C_6 - - C_6)$

8.25. $C_6 - - - C_6(C_6 - - C_6)$. By considering the 996 cases related to the existence of $C_6 - - C_6$ and 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$, it can be seen that there are 1594 cases for the relations of four C_6 cycles in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 1446 cases of these 1594 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 148 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 148 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(C_6 - - C_6)$.

- (1) $R_1 : h_2^3 h_3^{-1} h_2^{-1} h_3^2 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3^2 = 1, \\ R_4 : h_2^2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-2} (h_2^{-1} h_3)^3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -2)$ is solvable, a contradiction.
- (3) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (4) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, \\ R_4 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (7) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1$,
 $R_4 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (8) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (9) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (10) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (11) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (12) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (13) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (14) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (15) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (16) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (17) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (18) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (19) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.

- (20) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 5)$ is solvable, a contradiction.
- (21) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (22) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -5)$ is solvable, a contradiction.
- (23) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 5)$ is solvable, a contradiction.
- (24) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (25) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 5)$ is solvable, a contradiction.
- (26) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 5)$ is solvable, a contradiction.
- (27) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (28) $R_1 : h_2^2 h_3^{-1} h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(3, 1)$ is solvable, a contradiction.
- (29) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (30) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (31) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (32) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

- (33) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(4, 1)$ is solvable, a contradiction.
- (34) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(4, 1)$ is solvable, a contradiction.
- (35) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(4, 1)$ is solvable, a contradiction.
- (36) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (37) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (38) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (39) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (40) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (41) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (42) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (43) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (44) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (45) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.

- (46) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (47) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (48) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.
- (49) $R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2^3 h_3^3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (50) $R_1 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$,
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (51) $R_1 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$, $R_3 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (52) $R_1 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1$, $R_3 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (53) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (54) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (55) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (56) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (57) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (58) $R_1 : h_2 (h_3 h_2^{-2})^2 h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.

- (59) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$,
 $R_4 : h_2h_3^{-3}h_2h_3h_2^{-1}h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (60) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-2}h_3^2 = 1$,
 $R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (61) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-2}h_3^2 = 1$,
 $R_4 : (h_2h_3^{-1})^3(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (62) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$, $R_3 : h_2h_3^{-3}h_2^2h_3^{-1}h_2 = 1$,
 $R_4 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (63) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_3^{-1}h_2)^2 = 1$, $R_3 : h_2h_3^{-3}h_2^2h_3^{-1}h_2 = 1$,
 $R_4 : (h_2h_3^{-1})^3(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.
- (64) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2 = 1$,
 $R_4 : h_2(h_3^{-1}h_2h_3^{-1})^2h_2h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (65) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2 = 1$,
 $R_4 : h_2h_3^{-1}h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (66) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2(h_3h_2^{-2})^2h_3 = 1$,
 $R_4 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (67) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$,
 $R_4 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (68) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$,
 $R_4 : h_2h_3^{-1}h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (69) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-2}h_3 = 1$,
 $R_4 : h_2h_3^{-3}h_2^2h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (70) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2h_3 = 1$,
 $R_4 : h_2(h_3^{-1}h_2h_3^{-1})^2h_2h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.
- (71) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^{-1}h_2^{-1}h_3h_2^{-1}h_3^{-1}h_2h_3 = 1$,
 $R_4 : h_2h_3^{-1}h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.

$$(72) \quad R_1 : h_2 h_3 h_2^{-2} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-4, 1)$ is solvable, a contradiction.

$$(73) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(74) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1, \quad R_3 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(75) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^3 = 1, \\ R_4 : h_2 h_3^3 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(76) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^3 = 1, \\ R_4 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(77) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(78) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(79) \quad R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(80) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(81) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(82) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, \quad R_3 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(83) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(84) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(85) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(86) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(87) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(88) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(89) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(90) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(91) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(92) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(93) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(94) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.

$$(95) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.

$$(96) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(97) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(98) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(99) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-2})^2 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(100) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(101) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(102) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(103) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (33) and with the same discussion, there is a contradiction.

$$(104) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(105) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(106) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.

$$(107) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (39) and with the same discussion, there is a contradiction.

$$(108) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(109) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.

$$(110) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (32) and with the same discussion, there is a contradiction.

$$(111) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$$(112) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (30) and with the same discussion, there is a contradiction.

$$(113) \quad R_1 : h_2^2 h_3^{-2} (h_3^{-1} h_2)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (31) and with the same discussion, there is a contradiction.

$$(114) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (48) and with the same discussion, there is a contradiction.

$$(115) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(116) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(117) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(118) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (43) and with the same discussion, there is a contradiction.

$$(119) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-2})^2 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.

$$(120) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (40) and with the same discussion, there is a contradiction.

$$(121) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (41) and with the same discussion, there is a contradiction.

$$(122) \quad R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (44) and with the same discussion, there is a contradiction.

$$(123) \quad R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, \quad R_3 : h_2^3 h_3^3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (49) and with the same discussion, there is a contradiction.

$$(124) \quad R_1 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (50) and with the same discussion, there is a contradiction.

$$(125) \quad R_1 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3 (h_2 h_3^{-1} h_2)^2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (51) and with the same discussion, there is a contradiction.

$$(126) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, R_3 : (h_2 h_3)^2 h_2 h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(127) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (57) and with the same discussion, there is a contradiction.

$$(128) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (58) and with the same discussion, there is a contradiction.

$$(129) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (59) and with the same discussion, there is a contradiction.

$$(130) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (62) and with the same discussion, there is a contradiction.

$$(131) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (63) and with the same discussion, there is a contradiction.

$$(132) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (60) and with the same discussion, there is a contradiction.

$$(133) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (61) and with the same discussion, there is a contradiction.

$$(134) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (56) and with the same discussion, there is a contradiction.

$$(135) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (53) and with the same discussion, there is a contradiction.

$$(136) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^3 = 1:$$

By interchanging h_2 and h_3 in (54) and with the same discussion, there is a contradiction.

$$(137) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_3^{-1} h_2)^2 = 1, \quad R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \quad R_3 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^3 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (55) and with the same discussion, there is a contradiction.

$$(138) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2^2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (72) and with the same discussion, there is a contradiction.

$$(139) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 (h_2 h_3^{-2})^2 h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (69) and with the same discussion, there is a contradiction.

$$(140) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (70) and with the same discussion, there is a contradiction.

$$(141) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (71) and with the same discussion, there is a contradiction.

$$(142) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (67) and with the same discussion, there is a contradiction.

$$(143) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-2})^2 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (66) and with the same discussion, there is a contradiction.

$$(144) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (64) and with the same discussion, there is a contradiction.

$$(145) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3^{-3} h_2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (65) and with the same discussion, there is a contradiction.

$$(146) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} (h_2^{-1} h_3)^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (68) and with the same discussion, there is a contradiction.

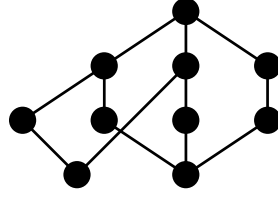
$$(147) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-1} h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (73) and with the same discussion, there is a contradiction.

$$(148) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : (h_2 h_3)^2 h_2^{-1} h_3^2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (74) and with the same discussion, there is a contradiction.

8.26. $\mathbf{C_6} - - - \mathbf{C_6}(-\mathbf{C_5}-)$. By considering the 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that there are 1482 cases for the relations of two cycles C_6 and a cycle C_5 in the graph $C_6 - - - C_6(-C_5-)$. Using

26) $C_6 - - - C_6(-C_5-)$

GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 1358 cases of these 1482 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 124 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(-C_5-)$ in $K(\alpha, \beta)$. In the following, we show that these 124 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_5-)$.

- (1) $R_1 : h_2^3 h_3^3 = 1, R_2 : h_2^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (2) $R_1 : h_2^3 h_3^3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2 h_3^2 = 1, R_3 : h_2^2 h_3^{-3} h_2 = 1:$
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (3) $R_1 : h_2^3 (h_3 h_2^{-1})^2 h_3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (4) $R_1 : h_2^2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3^2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1, R_2 : (h_2 h_3^{-1})^2 h_2 h_3^3 = 1, R_3 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (6) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$
 $\Rightarrow h_3 = 1$, a contradiction.
- (7) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (8) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1:$
 $\Rightarrow h_3 = 1$, a contradiction.
- (9) $R_1 : h_2^2 h_3 h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (10) $R_1 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (11) $R_1 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (12) $R_1 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (13) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (14) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (15) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (16) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (17) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^3 h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (18) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (19) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1, R_3 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (20) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (21) $R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (22) $R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (23) $R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (24) $R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ is solvable, a contradiction.
- (25) $R_1 : h_2^2 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-2} h_3 = 1:$
 By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (26) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (27) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow h_3 = 1$, a contradiction.
- (28) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (29) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow h_3 = 1$, a contradiction.
- (30) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (31) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (32) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (33) $R_1 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2 h_3^{-3} h_2 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$
 By interchanging h_2 and h_3 in (30) and with the same discussion, there is a contradiction.

- (34) $R_1 : h_2(h_2h_3^{-2})^2h_2 = 1, R_2 : h_2h_3h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1, R_3 : h_2h_3^2h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (35) $R_1 : h_2(h_2h_3^{-2})^2h_2 = 1, R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1, R_3 : h_2^2h_3h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (36) $R_1 : h_2^2h_3^{-1}h_2h_3h_2h_3^{-1}h_2 = 1, R_2 : h_2(h_2h_3^{-1})^2h_2^2h_3 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (37) $R_1 : h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3^2 = 1, R_2 : h_2h_3h_2^{-1}h_3h_2h_3^{-1}h_2h_3 = 1, R_3 : h_2^3h_3h_2^{-1}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (38) $R_1 : h_2^2h_3^{-1}h_2h_3h_2^{-1}h_3^2 = 1, R_2 : h_2h_3h_2^{-1}h_3h_2h_3^{-1}h_2h_3 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (39) $R_1 : h_2(h_2h_3^{-1})^2h_3^{-1}h_2h_3 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2^3h_3^{-2}h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (40) $R_1 : h_2^2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (41) $R_1 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1, R_2 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (42) $R_1 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (43) $R_1 : (h_2h_3)^2h_2^{-1}h_3^2 = 1, R_2 : (h_2h_3)^2h_2^{-1}h_3^2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 By interchanging h_2 and h_3 in (41) and with the same discussion, there is a contradiction.
- (44) $R_1 : (h_2h_3)^2h_2^{-1}h_3^2 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1:$
 By interchanging h_2 and h_3 in (42) and with the same discussion, there is a contradiction.
- (45) $R_1 : h_2h_3h_2h_3^{-1}(h_3^{-1}h_2)^2 = 1, R_2 : h_2h_3^{-2}h_2h_3h_2h_3^{-1}h_2 = 1, R_3 : h_2^3h_3^{-2}h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (46) $R_1 : h_2h_3h_2h_3^{-1}(h_3^{-1}h_2)^2 = 1, R_2 : h_2h_3^{-2}h_2h_3h_2h_3^{-1}h_2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (47) $R_1 : h_2h_3(h_2h_3^{-1}h_2)^2 = 1, R_2 : h_2h_3(h_2h_3^{-1}h_2)^2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (48) $R_1 : h_2h_3h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1, R_2 : h_2h_3^2h_2^{-1}h_3h_2h_3^{-1}h_2 = 1, R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1:$
 By interchanging h_2 and h_3 in (38) and with the same discussion, there is a contradiction.
- (49) $R_1 : h_2h_3h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1, R_2 : h_2h_3^2h_2^{-1}h_3h_2h_3^{-1}h_2 = 1, R_3 : h_2h_3^{-1}h_2h_3^3 = 1:$
 By interchanging h_2 and h_3 in (37) and with the same discussion, there is a contradiction.
- (50) $R_1 : h_2h_3h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2^3h_3h_2^{-1}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (51) $R_1 : h_2h_3h_2h_3^{-1}h_2h_3h_2^{-1}h_3 = 1, R_2 : h_2(h_3h_2^{-1})^2h_3^{-2}h_2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (52) $R_1 : h_2h_3(h_2h_3^{-1})^2h_2^{-1}h_3 = 1, R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (53) $R_1 : h_2h_3(h_2h_3^{-1})^2h_2^{-1}h_3 = 1, R_2 : (h_2h_3^{-1}h_2)^2h_3^2 = 1, R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1:$
 By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(54) R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(55) R_1 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.

$$(56) R_1 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(57) R_1 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(58) R_1 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(59) R_1 : h_2 h_3^2 h_2 h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(60) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(61) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(62) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(63) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(64) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(65) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.

$$(66) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (26) and with the same discussion, there is a contradiction.

$$(67) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.

$$(68) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.

$$(69) R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (39) and with the same discussion, there is a contradiction.

$$(70) R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^3 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(71) R_1 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.

$$(72) R_1 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (19) and with the same discussion, there is a contradiction.

$$(73) R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.

$$(74) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (46) and with the same discussion, there is a contradiction.

$$(75) \quad R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, R_3 : h_2 h_3^{-4} h_2 = 1:$$

By interchanging h_2 and h_3 in (45) and with the same discussion, there is a contradiction.

$$(76) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(77) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^3 h_3^2 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(78) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^3 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(79) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(80) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(81) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-1, 1)$ is solvable, a contradiction.

$$(82) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(83) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(84) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(85) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(86) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(87) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(88) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(89) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(90) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(91) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_3^2 (h_3 h_2^{-1})^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(92) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_3 (h_3 h_2^{-1})^3 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(93) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (31) and with the same discussion, there is a contradiction.

$$(94) \quad R_1 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (51) and with the same discussion, there is a contradiction.

$$(95) \quad R_1 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (50) and with the same discussion, there is a contradiction.

$$(96) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(97) \quad R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (52) and with the same discussion, there is a contradiction.

$$(98) \quad R_1 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \quad R_2 : h_2 (h_3 h_2^{-1} h_3)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (47) and with the same discussion, there is a contradiction.

$$(99) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(100) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (40) and with the same discussion, there is a contradiction.

$$(101) \quad R_1 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, \quad R_2 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (99) and with the same discussion, there is a contradiction.

$$(102) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (76) and with the same discussion, there is a contradiction.

$$(103) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^3 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (78) and with the same discussion, there is a contradiction.

$$(104) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 (h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (91) and with the same discussion, there is a contradiction.

$$(105) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^3 = 1:$$

By interchanging h_2 and h_3 in (77) and with the same discussion, there is a contradiction.

$$(106) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (84) and with the same discussion, there is a contradiction.

$$(107) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (85) and with the same discussion, there is a contradiction.

$$(108) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (89) and with the same discussion, there is a contradiction.

$$(109) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (86) and with the same discussion, there is a contradiction.

$$(110) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2^2 h_3^{-1} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (79) and with the same discussion, there is a contradiction.

$$(111) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 (h_2 h_3^{-1})^3 h_2 = 1:$$

By interchanging h_2 and h_3 in (92) and with the same discussion, there is a contradiction.

$$(112) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 h_3 (h_2 h_3^{-1})^2 h_2 = 1:$$

By interchanging h_2 and h_3 in (87) and with the same discussion, there is a contradiction.

$$(113) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, \quad R_3 : h_2 h_3^2 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (80) and with the same discussion, there is a contradiction.

$$(114) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (81) and with the same discussion, there is a contradiction.

$$(115) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (82) and with the same discussion, there is a contradiction.

$$(116) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (88) and with the same discussion, there is a contradiction.

$$(117) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (90) and with the same discussion, there is a contradiction.

$$(118) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (83) and with the same discussion, there is a contradiction.

$$(119) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(120) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1, R_3 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (32) and with the same discussion, there is a contradiction.

$$(121) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

$$(122) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3 = 1:$$

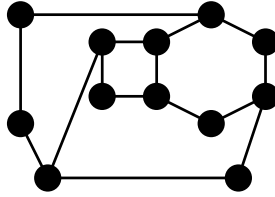
By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(123) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(124) \quad R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

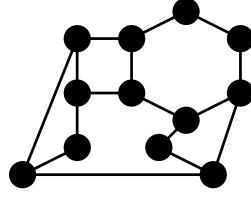


$$27) \quad C_4 - C_6(- - C_7 - -)(- - - C_6)$$

8.27. $C_4 - C_6(- - C_7 - -)(- - - C_6)$. By considering the 16 cases related to the existence of $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ from Table 18 and the relations from Table 5 which are not disproved, it can be seen that there are 124 cases for the relations of a cycle C_4 , two cycles C_6 and a cycle C_7 in the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 112 cases of these 124 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 12 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 12 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(- - - C_6)$.

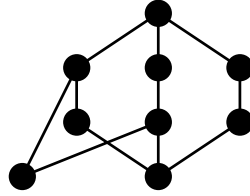
- (1) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-5} h_2 h_3 = 1$,
 $R_4 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (2) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2^2 h_3^{-1} (h_2^{-1} h_3)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (3) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1$,
 $R_4 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (4) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^4 h_3 = 1$,
 $R_4 : h_3^2 h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (5) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3^{-2})^2 h_2^{-1} h_3^2 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (6) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (7) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$, $R_3 : (h_2^2 h_3^{-1})^2 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.
- (8) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1$,
 $R_4 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$:
By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (9) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$, $R_3 : h_2^4 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$:
By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (10) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$, $R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2 h_3^{-2} (h_2^{-1} h_3)^3 = 1$:
By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (11) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$, $R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1$,
 $R_4 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (12) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$, $R_3 : (h_2 h_3^{-1})^4 h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2^2 h_3^{-1} (h_2 h_3^{-1} h_2)^2 = 1$:
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

8.28. $\mathbf{C}_4 - \mathbf{C}_6(- - \mathbf{C}_7 - -)(\mathbf{C}_4)(\mathbf{C}_4)$. By considering the 16 cases related to the existence of $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ from Table 18 and the relations from Table 2 which are not disproved, it can be seen that there are 8 cases for the relations of three cycles C_4 , a cycle C_7 and a



28) $C_4 - C_6(- - C_7 - -)(C_4)(C_4)$

cycle C_6 in this structure. By considering all groups with two generators h_2 and h_3 and five relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_4)(C_4)$.



29) $C_6 - - - C_6(-C_5 - -)$

8.29. $C_6 - - - C_6(-C_5 - -)$. By considering the 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that there are 418 cases for the relations of two cycles C_6 and a cycle C_5 in the graph $C_6 - - - C_6(-C_5 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 358 cases of these 418 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 60 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - - C_6(-C_5 - -)$ in $K(\alpha, \beta)$. In the following, we show that these 60 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_5 - -)$.

- (1) $R_1 : h_2^3 h_3^3 = 1, R_2 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1, R_3 : h_2^2 h_3^{-3} h_2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (2) $R_1 : h_2^3 h_3^3 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2 = 1:$
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (3) $R_1 : h_2^2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (4) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^3 h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2^3 h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $R_1 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3)^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ is solvable, a contradiction.

- (7) $R_1 : h_2^2(h_3h_2^{-1}h_3)^2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_3 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (8) $R_1 : h_2^2(h_3h_2^{-1}h_3)^2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_3 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (9) $R_1 : h_2^2(h_3h_2^{-1}h_3)^2 = 1$, $R_2 : h_2(h_3h_2^{-1})^2h_3^{-1}h_2h_3 = 1$, $R_3 : h_2h_3^2h_2^{-1}h_3^2 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (10) $R_1 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1$, $R_2 : h_2h_3h_2h_3^{-1}(h_2^{-1}h_3)^2 = 1$, $R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (11) $R_1 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_3 : h_2h_3^{-1}h_2(h_3h_2^{-1})^2h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (12) $R_1 : h_2^2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-1}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_3 : (h_2h_3^{-1})^2(h_2^{-1}h_3)^2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (13) $R_1 : h_2^2h_3^{-2}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3h_2h_3^{-2}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (14) $R_1 : h_2^2h_3^{-2}h_2h_3^{-1}h_2^{-1}h_3 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-1}h_3^{-1}h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (15) $R_1 : h_2(h_2h_3^{-2})^2h_2 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1$, $R_3 : h_2(h_3^{-1}h_2h_3^{-1})^2h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (16) $R_1 : h_2(h_2h_3^{-2})^2h_2 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3h_2h_3^{-2}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (17) $R_1 : h_2(h_2h_3^{-2})^2h_2 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-1}h_3^{-1}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (18) $R_1 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1$, $R_2 : (h_2h_3)^2h_2h_3^{-1}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^{-2}h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (19) $R_1 : (h_2h_3)^2h_2^{-1}h_3^2 = 1$, $R_2 : (h_2h_3)^2h_2^{-1}h_3^2 = 1$, $R_3 : h_2h_3^{-2}h_2^{-1}h_3^2 = 1$:
By interchanging h_2 and h_3 in (18) and with the same discussion, there is a contradiction.
- (20) $R_1 : h_2h_3(h_2h_3^{-1})^2h_2^{-1}h_3 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2h_3^{-1}h_2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^2h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (21) $R_1 : h_2h_3(h_2h_3^{-1})^2h_2^{-1}h_3 = 1$, $R_2 : (h_2h_3^{-1}h_2)^2h_3^2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^2h_3 = 1$:
By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.
- (22) $R_1 : h_2h_3(h_2h_3^{-1})^2h_2h_3 = 1$, $R_2 : (h_2h_3^{-1}h_2)^2h_3^2 = 1$, $R_3 : h_2^2h_3^{-1}h_2^2h_3 = 1$:
By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (23) $R_1 : h_2h_3^2h_2^{-1}h_3^{-1}h_2^{-1}h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-3}h_2 = 1$, $R_3 : h_2h_3^{-4}h_2 = 1$:
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (24) $R_1 : h_2(h_3h_2^{-2})^2h_3 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-2}h_3 = 1$, $R_3 : h_2h_3h_2^{-1}h_3^{-2}h_2 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (25) $R_1 : h_2h_3h_2^{-2}h_3h_2^{-1}h_3^{-1}h_2 = 1$, $R_2 : h_2h_3^{-1}h_2^{-1}h_3^2h_2^{-2}h_3 = 1$, $R_3 : h_2h_3h_2^{-1}h_3^{-1}h_2^{-1}h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (26) $R_1 : h_2h_3h_2^{-1}(h_2^{-1}h_3)^2h_3 = 1$, $R_2 : h_2h_3h_2^{-1}h_3^{-3}h_2 = 1$, $R_3 : h_2h_3^{-4}h_2 = 1$:
By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

- (27) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (28) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (29) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (30) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (31) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1$:
 $\Rightarrow G$ has a torsion element, a contradiction.
- (32) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow G$ is solvable, a contradiction.
- (33) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (34) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (35) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.
- (36) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.
- (37) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 h_3 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1$, $R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.
- (38) $R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2 = 1$:
By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.
- (39) $R_1 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2 h_3^2 h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (20) and with the same discussion, there is a contradiction.
- (40) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (29) and with the same discussion, there is a contradiction.
- (41) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1$:
By interchanging h_2 and h_3 in (30) and with the same discussion, there is a contradiction.
- (42) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^{-3} h_2 h_3 = 1$:
By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.
- (43) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1$, $R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.
- (44) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$:
By interchanging h_2 and h_3 in (28) and with the same discussion, there is a contradiction.
- (45) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-1} h_2 = 1$, $R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1$, $R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1$:
By interchanging h_2 and h_3 in (27) and with the same discussion, there is a contradiction.
- (46) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^{-2} h_2 h_3^{-1} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (25) and with the same discussion, there is a contradiction.

$$(47) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, R_2 : (h_2 h_3^{-2})^2 h_2 h_3 = 1, R_3 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(48) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, R_3 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (35) and with the same discussion, there is a contradiction.

$$(49) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-2} h_3^2 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

By interchanging h_2 and h_3 in (36) and with the same discussion, there is a contradiction.

$$(50) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (32) and with the same discussion, there is a contradiction.

$$(51) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (33) and with the same discussion, there is a contradiction.

$$(52) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1:$$

By interchanging h_2 and h_3 in (31) and with the same discussion, there is a contradiction.

$$(53) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

By interchanging h_2 and h_3 in (34) and with the same discussion, there is a contradiction.

$$(54) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-3} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(55) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-3} h_2^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(56) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(57) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(58) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : h_2^2 h_3^{-2} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(59) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$$

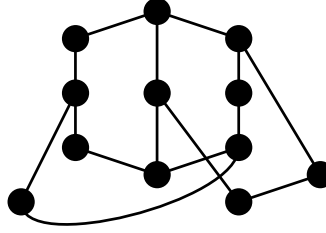
By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(60) R_1 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$C_6 - -C_6(- -C_5-)$ subgraph: By considering the 996 cases related to the existence of $C_6 - -C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that there are 3267 cases for the relations of two cycles C_6 and a cycle C_5 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 2910 cases of these 3267 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 357 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(- -C_5-)$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 349 cases of these relations lead to contradictions and 8 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(- -C_5-)$ in the graph $K(\alpha, \beta)$.

8.30. **$C_6 - -C_6(- -C_5-)(-C_5-)$.** By considering the 8 cases related to the existence of $C_6 - -C_6(- -C_5-)$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that

30) $C_6 - - C_6(- - C_5-)(-C_5-)$

there are 62 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 56 cases of these 62 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - - C_6(- - C_5-)(-C_5-)$ in $K(\alpha, \beta)$. In the following, we show that these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - C_6(- - C_5-)(-C_5-)$.

$$(1) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(2) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, \\ R_4 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(3) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, \\ R_4 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(4) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1:$$

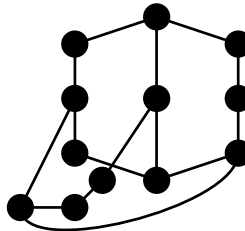
By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(5) R_1 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(6) R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

31) $C_6 - - C_6(- - C_5-)(C_6 - - -)$

8.31. $\mathbf{C_6} - -\mathbf{C_6}(- -\mathbf{C_5-})(\mathbf{C_6} - - -)$. By considering the 8 cases related to the existence of $C_6 - -C_6(- -C_5-)$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 76 cases for the relations of a cycle C_5 and three cycles C_6 in the graph $C_6 - -C_6(- -C_5-)(C_6 - - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 64 cases of these 76 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 12 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(- -C_5-)(C_6 - - -)$ in $K(\alpha, \beta)$. In the following, we show that these 12 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - -C_6(- -C_5-)(C_6 - - -)$.

$$(1) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3 h_2^{-2} h_3 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(2) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : (h_2 h_3^{-1})^2 h_2 h_3^2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^2 = 1:$$

$\Rightarrow h_2 = 1$, a contradiction.

$$(3) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, \\ R_4 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(4) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : (h_2 h_3^{-1})^3 h_2 h_3 = 1, \\ R_4 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

$$(5) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, \\ R_4 : h_2^2 (h_3 h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(6) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2^2 (h_3 h_2^{-1})^2 h_3 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(7) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, \\ R_4 : h_2^2 h_3^{-2} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(8) \ R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 (h_3 h_2^{-1})^3 h_3 = 1, \\ R_4 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(9) \ R_1 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1, \\ R_4 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow h_2 = 1$, a contradiction.

$$(10) \ R_1 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-2} h_2 h_3^2 = 1, \\ R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 h_3 = 1:$$

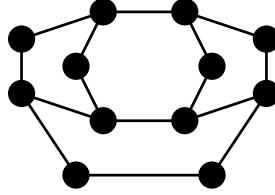
$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

$$(11) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1, \\ R_4 : (h_2 h_3)^2 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(12) \quad R_1 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_2 : (h_2 h_3^{-1})^2 h_2^{-1} h_3^2 h_2^{-1} h_3 = 1, \quad R_3 : h_2^2 h_3 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3 h_2 h_3^{-1} h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.



32) $C_5(- - C_6 - -)C_5(C_6)$

8.32. $C_5(- - C_6 - -)C_5(C_6)$. By considering the 6 cases related to the existence of $C_5(- - C_6 - -)C_5$ in the graph $K(\alpha, \beta)$ and the relations from Table 5 which are not disproved, it can be seen that there are 120 cases for the relations of two cycles C_5 and two cycles C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 104 cases of these 120 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_6)$ in $K(\alpha, \beta)$. In the following, we show that these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_6)$.

$$(1) \quad R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_2 h_3^{-1})^2 h_2^{-2} h_3 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(2) \quad R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

$\Rightarrow h_2 = 1$, a contradiction.

$$(3) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(4) \quad R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \quad R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1})^2 h_3^{-2} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(5) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, \quad R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (6) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1)$ is solvable, a contradiction.

- (7) $R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 (h_2 h_3^{-1})^2 h_2^{-2} h_3 = 1$:

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

- (8) $R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$, $R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3^{-1} h_2 = 1$:

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

- (9) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$,
 $R_4 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

- (10) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1$,
 $R_4 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.

- (11) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

- (12) $R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$, $R_3 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

- (13) $R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1$:

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

- (14) $R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1$:

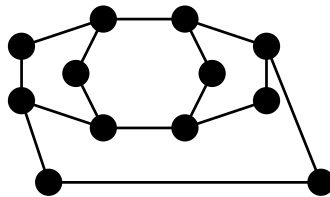
By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

- (15) $R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1$:

By interchanging h_2 and h_3 in (10) and with the same discussion, there is a contradiction.

- (16) $R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1$, $R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1$, $R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1$:

By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.



33) $C_5(- - C_6 - -)C_5(C_7)$

8.33. $\mathbf{C_5(- - C_6 - -)C_5(C_7)}$. By considering the 6 cases related to the existence of $C_5(- - C_6 - -)C_5$ in the graph $K(\alpha, \beta)$ and the relations of C_7 cycles, it can be seen that there are 248 cases for the relations of two cycles C_5 , a cycle C_6 and a cycle C_7 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 220 cases of these 248 cases are finite and solvable, or just finite, that is a contradiction with the assumptions. So, there are just 28 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_7)$ in $K(\alpha, \beta)$. In the following, we show that these 28 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5(- - C_6 - -)C_5(C_7)$.

$$(1) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2^2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-5, 1)$ is solvable, a contradiction.

$$(2) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2^2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.

$$(3) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.

$$(4) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3^{-2} h_2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(5, 1)$ is solvable, a contradiction.

$$(5) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 h_3^{-1} (h_2^{-1} h_3)^4 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(6) \ R_1 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 h_3^{-1} (h_3^{-1} h_2)^4 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(7) \ R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(8) \ R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-3} h_2 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(9) \ R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(10) \ R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, \ R_2 : h_2 h_3^{-3} h_2 h_3 = 1, \ R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(11) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-2} h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(12) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-3} h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(13) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1) \text{ is solvable, a contradiction.}$$

$$(14) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(15) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 h_3^{-1} (h_2^{-1} h_3)^4 = 1: \\ \Rightarrow G \text{ has a torsion element, a contradiction.}$$

$$(16) \quad R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2 h_3^{-1} (h_3^{-1} h_2)^4 = 1: \\ \Rightarrow G \text{ has a torsion element, a contradiction.}$$

$$(17) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2^2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (11) and with the same discussion, there is a contradiction.

$$(18) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2^2 (h_3^{-1} h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (12) and with the same discussion, there is a contradiction.

$$(19) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (13) and with the same discussion, there is a contradiction.

$$(20) \quad R_1 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2 (h_3 h_2^{-1})^2 h_3^{-1} h_2 h_3^{-2} h_2 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.

$$(21) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 1) \text{ is solvable, a contradiction.}$$

$$(22) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 (h_2 h_3^{-1})^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1} h_2)^2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1) \text{ is solvable, a contradiction.}$$

$$(23) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1, \\ R_4 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2 h_3^{-2} h_2 = 1: \\ \Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1) \text{ is solvable, a contradiction.}$$

$$(24) \quad R_1 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 h_3 h_2^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2^3 h_3^{-2} h_2^{-1} h_3 = 1,$$

$$R_4 : (h_2 h_3^{-1})^3 h_3^{-1} h_2 h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(25) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1,$$

$$R_4 : h_2^2 (h_3 h_2^{-1})^3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (24) and with the same discussion, there is a contradiction.

$$(26) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_3 : h_2^2 h_3^{-3} h_2^{-1} h_3 = 1,$$

$$R_4 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (23) and with the same discussion, there is a contradiction.

$$(27) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1,$$

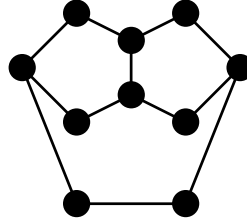
$$R_4 : (h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (22) and with the same discussion, there is a contradiction.

$$(28) \quad R_1 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 h_3^{-1} h_2^{-1} h_3 = 1, R_3 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1,$$

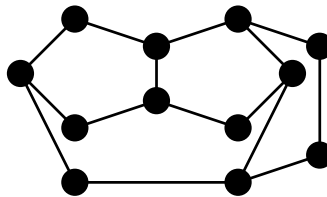
$$R_4 : h_2 h_3^{-1} (h_2^{-1} h_3)^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (21) and with the same discussion, there is a contradiction.



$C_5 - C_5(- - C_7 - -)$

$C_5 - C_5(- - C_7 - -)$ subgraph: By considering the relations from Table 21 which are not disproved and the relations of C_7 cycles, it can be seen that there are 1981 cases for the relations of two cycles C_5 and a cycle C_7 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 1785 cases of these 1981 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 196 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 177 cases of these relations lead to contradictions and 19 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)$ in the graph $K(\alpha, \beta)$.



34) $C_5 - C_5(- - C_7 - -)(- - C_5)$

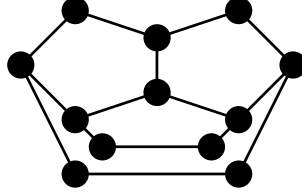
8.34. $\mathbf{C_5 - C_5(- - C_7 - -)(- - C_5)}$. By considering the 19 cases related to the existence of $C_5 - C_5(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that there are 256 cases for the relations of a cycle C_7 and three cycles C_5 in the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 232 cases of these 256 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 24 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$ in $K(\alpha, \beta)$. In the following, we show that these 24 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(- - C_5)$.

- (1) $R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, R_3 : (h_2 h_3)^2 (h_3 h_2^{-1})^2 h_3 = 1,$
 $R_4 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ has a torsion element, a contradiction.
- (2) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1,$
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow G$ is solvable, a contradiction.
- (3) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-3} h_2^{-1} h_3 = 1,$
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, 2)$ is solvable, a contradiction.
- (4) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-3} h_2^{-1} h_3 = 1,$
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} h_2 h_3^2 = 1,$
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow G$ is solvable, a contradiction.
- (6) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-2} h_2 h_3 h_2^{-1} h_3 = 1,$
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1:$
 $\Rightarrow G$ is solvable, a contradiction.
- (7) $R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1,$
 $R_4 : (h_2 h_3^{-1} h_2)^2 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (8) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-3} h_2 = 1,$
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^2 h_2^{-1} h_3^{-3} h_2 = 1,$
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(-2, 1)$ is solvable, a contradiction.
- (10) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3 h_2^{-1} h_3^{-3} h_2 = 1,$
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

- (11) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$,
 $R_4 : h_2 h_3^{-3} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (12) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$,
 $R_4 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (13) $R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_2 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1$, $R_3 : h_2^2 h_3 h_2^{-1} h_3^{-3} h_2 = 1$,
 $R_4 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (14) $R_1 : h_2 h_3 h_2 h_3^{-2} h_2 = 1$, $R_2 : h_2 h_3 h_2 h_3^{-2} h_2 = 1$, $R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(2, 1)$ is solvable, a contradiction.
- (15) $R_1 : h_2 h_3 h_2^{-2} h_3^2 = 1$, $R_2 : h_2 h_3 h_2^{-2} h_3^2 = 1$, $R_3 : h_2 h_3 h_2 h_3^{-2} h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.
- (16) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2^3 (h_3^{-1} h_2^{-1})^2 h_3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (17) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2^3 (h_3^{-1} h_2^{-1})^2 h_3 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$:
By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (18) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 (h_2 h_3)^2 h_2^{-2} h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.
- (19) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1$:
By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (20) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3)^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.
- (21) $R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3 h_2^{-2} h_3 h_2 h_3^{-1} h_2 h_3 = 1$,
 $R_4 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1$:
By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (22) $R_1 : h_2 h_3^{-3} h_2 h_3 = 1$, $R_2 : h_2 h_3^{-3} h_2 h_3 = 1$, $R_3 : h_2^2 h_3 h_2^{-2} h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1$:
By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.
- (23) $R_1 : h_2 h_3^{-3} h_2 h_3 = 1$, $R_2 : h_2 h_3^{-3} h_2 h_3 = 1$, $R_3 : h_2^2 h_3 h_2^{-2} h_3^{-2} h_2 = 1$,
 $R_4 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1$:
By interchanging h_2 and h_3 in (9) and with the same discussion, there is a contradiction.

$$(24) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, \quad R_3 : h_2 (h_2 h_3^{-1})^2 (h_2 h_3)^2 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.



$$35) \quad C_5 - C_5(- - C_7 - -)(-C_5-)$$

8.35. $C_5 - C_5(- - C_7 - -)(-C_5-)$. By considering the 19 cases related to the existence of $C_5 - C_5(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ and the relations from Table 4 which are not disproved, it can be seen that there are 138 cases for the relations of a cycle C_7 and three cycles C_5 in the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 132 cases of these 138 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$ in $K(\alpha, \beta)$. In the following, we show that these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(- - C_7 - -)(-C_5-)$.

$$(1) \quad R_1 : h_2^2 h_3 h_2^{-2} h_3 = 1, \quad R_2 : h_2^2 h_3 h_2^{-2} h_3 = 1, \quad R_3 : (h_2 h_3)^2 (h_3 h_2^{-1})^2 h_3 = 1, \\ R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

$\Rightarrow G$ has a torsion element, a contradiction.

$$(2) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-2} (h_2^{-1} h_3)^2 = 1, \\ R_4 : h_2 h_3^2 h_2^{-1} h_3^2 = 1:$$

$\Rightarrow G$ is solvable, a contradiction.

$$(3) \quad R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, \quad R_3 : h_2 h_3 h_2 h_3^{-2} h_2 h_3^2 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2 h_3^2 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(4) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2 (h_2 h_3)^2 h_2^{-2} h_3 = 1, \\ R_4 : h_2^2 h_3^2 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

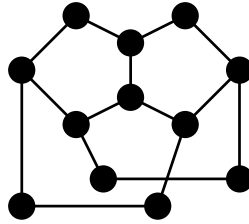
$$(5) \quad R_1 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, \quad R_3 : h_2^2 (h_3^{-1} h_2^{-1})^2 h_3 h_2^{-1} h_3 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(6) \quad R_1 : h_2 h_3^{-2} h_2 h_3^2 = 1, \quad R_2 : h_2 h_3^{-2} h_2 h_3^2 = 1, \quad R_3 : h_2 (h_2 h_3^{-1})^2 (h_2 h_3)^2 = 1, \\ R_4 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

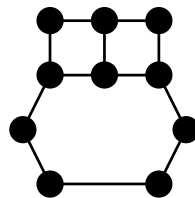
By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

8.36. $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$. By considering the relations from Tables 23 and 5 which are not disproved, it can be seen that there are 22 cases for the relations of two cycles C_5 and two cycles



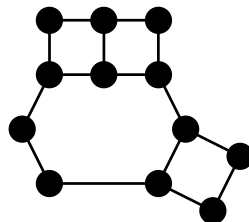
36) $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$

C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_6 - 2)$.



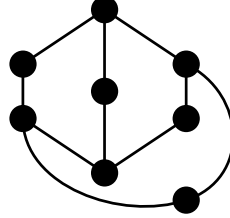
$C_4 - C_4(-C_7 -)$

$C_4 - C_4(-C_7 -)$ subgraph: By considering the possible cases of $C_4 - C_4$ from Remark 4.4 and the relations of C_7 cycles, it can be seen that there are 258 cases for the relations of two cycles C_4 and a cycle C_7 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 236 cases of these 258 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 22 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_4(-C_7 -)$ in $K(\alpha, \beta)$. Similar to the previous mentioned subgraphs it can be seen that 16 cases of these relations lead to contradictions and 6 cases of them may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_4(-C_7 -)$ in the graph $K(\alpha, \beta)$.



37) $C_4 - C_4(-C_7 -)(C_4)$

8.37. **$C_4 - C_4(-C_7 -)(C_4)$.** By considering the 6 cases related to the existence of $C_4 - C_4(-C_7 -)$ in the graph $K(\alpha, \beta)$ and the relations from Table 2 which are not disproved, it can be seen that there are 32 cases for the relations of a cycle C_7 and three cycles C_4 in the graph $C_4 - C_4(-C_7 -)(C_4)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are finite and solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_4(-C_7 -)(C_4)$.

38) $C_5 - - C_5(-C_5 - -)$

8.38. $C_5 - - C_5(-C_5 - -)$. By considering the relations from Tables 13 and 4 which are not disproved, it can be seen that there are 64 cases for the relations of three cycles C_5 in the graph $C_5 - - C_5(-C_5 - -)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 58 cases of these 64 cases are finite and solvable, that is a contradiction with the assumptions. So, there are just 6 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_5 - - C_5(-C_5 - -)$ in $K(\alpha, \beta)$. In the following, we show that these 6 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - - C_5(-C_5 - -)$.

$$(1) R_1 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1, R_2 : (h_2 h_3)^2 h_2^{-1} h_3 = 1, R_3 : h_2^2 h_3^2 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(2) R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-3} h_2 h_3 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.

$$(3) R_1 : h_2^2 h_3^{-2} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} (h_3^{-1} h_2)^3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(-3, 1)$ is solvable, a contradiction.

$$(4) R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1, R_3 : h_2^2 h_3^{-1} h_2 h_3^2 = 1:$$

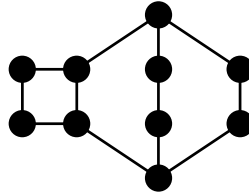
By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(5) R_1 : h_2 (h_3 h_2^{-1} h_3)^2 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.

$$(6) R_1 : (h_2 h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^2 (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

39) $C_6 - - - C_6(-C_4)$

8.39. $C_6 - - - C_6(-C_4)$. By considering the 99 cases related to the existence of $C_6 - - - C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 2 which are not disproved, it can be seen that there are 420 cases for the relations of two cycles C_6 and a cycle C_4 in the graph $C_6 - - - C_6(-C_4)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 398 cases of these 420 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 22 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the

graph $C_6 - - - C_6(-C_4)$ in $K(\alpha, \beta)$. In the following, we show that these 22 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - - - C_6(-C_4)$.

- (1) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (2) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2 h_3^{-2} h_2^2 h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (4) $R_1 : h_2^2 h_3^{-1} h_2 h_3 h_2^{-1} h_3^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (6) $R_1 : h_2 h_3 h_2 h_3^{-1} (h_3^{-1} h_2)^2 = 1, R_2 : h_2 h_3^{-2} h_2 h_3 h_2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (7) $R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} h_3 h_2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$:
 By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.
- (8) $R_1 : h_2 h_3 h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.
- (10) $R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.
- (11) $R_1 : h_2 h_3 h_2^{-2} h_3^2 h_2^{-1} h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.
- (12) $R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (13) $R_1 : h_2 h_3 h_2^{-1} (h_2^{-1} h_3)^2 h_3 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^2 h_2^{-2} h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1$:
 By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.
- (14) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (15) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-3} h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (16) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (17) $R_1 : h_2 h_3 h_2^{-1} h_3^{-1} h_2 h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.
- (18) $R_1 : h_2 h_3 h_2^{-1} h_3 h_2 h_3^{-1} h_2 h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_3^{-2} h_2 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1$:
 By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.
- (19) $R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3 h_2^{-1} h_3 = 1$:
 By interchanging h_2 and h_3 in (16) and with the same discussion, there is a contradiction.

$$(20) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2^2 h_3^{-2} h_2 = 1:$$

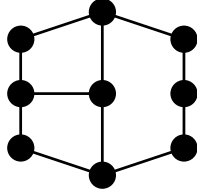
By interchanging h_2 and h_3 in (15) and with the same discussion, there is a contradiction.

$$(21) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : h_2 (h_3 h_2^{-1})^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (17) and with the same discussion, there is a contradiction.

$$(22) R_1 : h_2 h_3^{-1} h_2^{-1} h_3^2 h_2^{-1} h_3^{-1} h_2 = 1, R_2 : h_2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_3 : (h_2 h_3^{-1})^2 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (14) and with the same discussion, there is a contradiction.



40) $C_6 - -C_6(C_4)$

8.40. $C_6 - -C_6(C_4)$. By considering the 996 cases related to the existence of $C_6 - -C_6$ in the graph $K(\alpha, \beta)$ and the relations from Table 2 which are not disproved, it can be seen that there are 279 cases for the relations of two cycles C_6 and a cycle C_4 in the graph $C_6 - -C_6(C_4)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 268 cases of these 279 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 11 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_6 - -C_6(C_4)$ in $K(\alpha, \beta)$. In the following, we show that these 11 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_6 - -C_6(C_4)$.

$$(1) R_1 : h_2^3 h_3^3 = 1, R_2 : h_2^2 h_3^{-1} h_2 h_3^2 h_2^{-1} h_3 = 1, R_3 : h_2 h_3^{-1} h_2 h_3 h_2^{-1} h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(2) R_1 : h_2^2 (h_2 h_3^{-1})^3 h_2 = 1, R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle = \langle h_2 \rangle$ is abelian, a contradiction.

$$(3) R_1 : h_2^2 h_3 h_2^{-1} h_3^{-2} h_2 = 1, R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(4) R_1 : h_2^2 h_3^{-1} h_2^{-1} h_3^{-1} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(5) R_1 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(6) R_1 : h_2 (h_2 h_3^{-1})^2 h_3^{-1} h_2 h_3 = 1, R_2 : (h_2 h_3^{-1})^3 h_2^{-1} h_3^2 = 1, R_3 : h_2 h_3^{-2} h_2 h_3 = 1:$$

$\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.

$$(7) R_1 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 h_3 h_2^{-1} h_3^{-3} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(8) R_1 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(9) R_1 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_2 : h_3^2 (h_3 h_2^{-1})^3 h_3 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1:$$

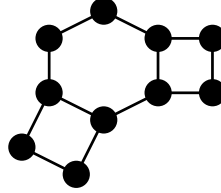
By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.

$$(10) R_1 : h_2 h_3^2 h_2^{-1} h_3^{-1} h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

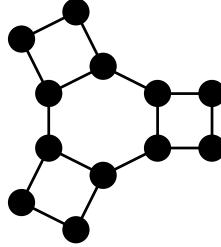
$$(11) R_1 : h_2 h_3 (h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^3 h_3^{-1} h_2 = 1, R_3 : h_2 h_3 h_2^{-2} h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.



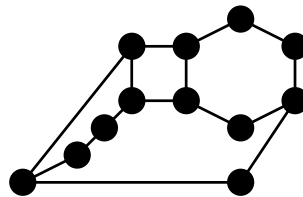
$C_4 - C_6(-C_4)$

$C_4 - C_6(-C_4)$ subgraph: By considering the relations from Tables 16 and 2 which are not disproved, it can be seen that there are 94 cases for the relations of two cycles C_4 and a cycle C_6 in this structure. Using GAP [9], we see that all groups with two generators h_2 and h_3 and three relations which are between 84 cases of these 94 cases are finite or solvable, that is a contradiction with the assumptions. So, there are just 10 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(-C_4)$ in $K(\alpha, \beta)$.



41) $C_4 - C_6(-C_4)(-C_4)$

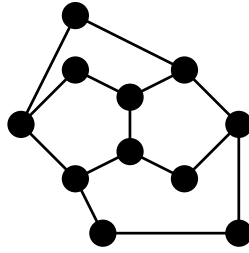
8.41. **$C_4 - C_6(-C_4)(-C_4)$.** By considering the 10 cases related to the existence of $C_4 - C_6(-C_4)$ in the graph $K(\alpha, \beta)$ and the relations from Table 2 which are not disproved, it can be seen that there are 36 cases for the relations of a cycle C_6 and three cycles C_4 in the graph $C_4 - C_6(-C_4)(-C_4)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(-C_4)(-C_4)$.



42) $C_4 - C_6(- - C_7 - -)(-C_5-)$

8.42. $\mathbf{C_4 - C_6(- - C_7 - -)(-C_5-)}$. By considering the 16 cases related to the existence of $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ from Table 18 and the relations from Table 4 which are not disproved, it can be seen that there are 62 cases for the relations of a cycle C_4 , a cycle C_6 , a cycle C_7 and a cycle C_5 in the graph $C_4 - C_6(- - C_7 - -)(-C_5-)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 58 cases of these 62 cases are solvable, that is a contradiction with the assumptions. So, there are just 4 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(-C_5-)$ in $K(\alpha, \beta)$. In the following, we show that these 4 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(-C_5-)$.

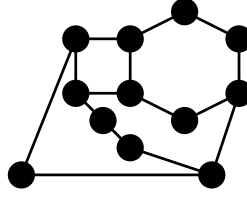
- (1) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1, R_3 : h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1,$
 $R_4 : h_2^2 h_3^{-1} h_2^{-2} h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (2) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1, R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1, R_3 : (h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1,$
 $R_4 : h_2 (h_3 h_2^{-1})^3 h_3 = 1:$
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, R_3 : h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1} h_2)^2 = 1,$
 $R_4 : (h_2 h_3^{-1})^3 h_2 h_3 = 1:$
 By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.
- (4) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1, R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1,$
 $R_4 : h_2 h_3^{-2} h_2^{-1} h_3^2 = 1:$
 By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.



43) $\mathbf{C_5 - C_5(-C_6 - -)(- - C_5-)}$

8.43. $\mathbf{C_5 - C_5(-C_6 - -)(- - C_5-)}$. By considering the relations from Tables 23 and 4 which are not disproved, it can be seen that there are 14 cases for the relations of three cycles C_5 and a cycle C_6 in the graph $C_5 - C_5(-C_6 - -)(- - C_5-)$. By considering all groups with two generators h_2 and h_3 and four relations which are between these cases and by using GAP [9], we see that all of these groups are solvable. So, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_5 - C_5(-C_6 - -)(- - C_5-)$.

8.44. $\mathbf{C_4 - C_6(- - C_7 - -)(C_7 - 2)}$. By considering the 16 cases related to the existence of $C_4 - C_6(- - C_7 - -)$ in the graph $K(\alpha, \beta)$ from Table 18 and the relations of C_7 cycles, it can be seen that there are 168 cases for the relations of a cycle C_4 , a cycle C_6 and two cycles C_7 in the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$. Using GAP [9], we see that all groups with two generators h_2 and h_3 and four relations which are between 152 cases of these 168 cases are solvable, that is a contradiction



$$44) C_4 - C_6(- - C_7 - -)(C_7 - 2)$$

with the assumptions. So, there are just 16 cases for the relations of these cycles which may lead to the existence of a subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$ in $K(\alpha, \beta)$. In the following, we show that these 16 cases lead to contradictions and so, the graph $K(\alpha, \beta)$ contains no subgraph isomorphic to the graph $C_4 - C_6(- - C_7 - -)(C_7 - 2)$.

- (1) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-4} h_2 h_3 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-4} (h_3^{-1} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (2) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 h_3^2 h_2^{-1} (h_2^{-1} h_3)^2 = 1$, $R_3 : h_2 h_3^{-2} h_2 (h_3 h_2^{-1})^3 h_3 = 1$,
 $R_4 : h_2 h_3 (h_3 h_2^{-1})^4 h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (3) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3^{-2})^2 h_2^{-1} h_3^2 = 1$,
 $R_4 : (h_2 h_3^{-3})^2 h_2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (4) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-2} (h_2^{-1} h_3^2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (5) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : (h_2 h_3^{-2})^2 h_2 h_3^{-1} h_2^{-1} h_3 = 1$,
 $R_4 : h_2 h_3^{-2} (h_3^{-2} h_2)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (6) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 (h_3 h_2^{-1} h_3)^2 h_2^{-1} h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (7) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3^{-1} (h_2^{-1} h_3^2)^2 h_2^{-1} h_3 = 1$,
 $R_4 : h_2 (h_3^{-1} h_2 h_3^{-1})^2 h_2^{-1} h_3 h_2^{-1} h_3 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle \cong BS(1, -1)$ is solvable, a contradiction.
- (8) $R_1 : h_2 h_3 h_2^{-2} h_3 = 1$, $R_2 : h_2 (h_3 h_2^{-1})^2 h_2^{-1} h_3^2 = 1$, $R_3 : h_2 h_3^{-1} h_2 h_3^{-1} h_2^{-1} (h_3 h_2^{-1} h_3)^2 = 1$,
 $R_4 : h_2 ((h_3 h_2^{-1})^2 h_3)^2 = 1$:
 $\Rightarrow \langle h_2, h_3 \rangle = \langle h_3 \rangle$ is abelian, a contradiction.
- (9) $R_1 : h_2 h_3^{-2} h_2 h_3 = 1$, $R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1$, $R_3 : (h_2^2 h_3^{-1})^2 h_2^{-2} h_3 = 1$,
 $R_4 : h_2^2 h_3^{-2} h_2^3 h_3^{-1} h_2 = 1$:
By interchanging h_2 and h_3 in (3) and with the same discussion, there is a contradiction.

$$(10) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1} h_2)^2 = 1, \\ R_4 : h_2^3 h_3^{-2} h_2^2 h_3^{-1} h_2 = 1:$$

By interchanging h_2 and h_3 in (5) and with the same discussion, there is a contradiction.

$$(11) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2^{-1} h_3 (h_2 h_3^{-1} h_2)^2 = 1, \\ R_4 : h_2^2 h_3^{-1} (h_2^{-2} h_3)^2 = 1:$$

By interchanging h_2 and h_3 in (4) and with the same discussion, there is a contradiction.

$$(12) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3 = 1:$$

By interchanging h_2 and h_3 in (7) and with the same discussion, there is a contradiction.

$$(13) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} (h_2^{-1} h_3 h_2^{-1})^2 h_2^{-1} h_3 = 1, \\ R_4 : (h_2 h_3^{-1} h_2)^2 h_3^{-1} h_2 h_3^{-1} h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (6) and with the same discussion, there is a contradiction.

$$(14) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2 h_3 (h_2 h_3^{-1})^2 h_3^{-1} h_2 = 1, \quad R_3 : h_2 h_3^{-1} h_2 (h_3 h_2^{-1})^2 (h_3^{-1} h_2)^2 = 1, \\ R_4 : ((h_2 h_3^{-1})^2 h_2)^2 h_3 = 1:$$

By interchanging h_2 and h_3 in (8) and with the same discussion, there is a contradiction.

$$(15) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2^3 h_3^{-1} h_2^{-1} h_3 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2^4 h_3^{-1} (h_3^{-1} h_2)^2 = 1:$$

By interchanging h_2 and h_3 in (1) and with the same discussion, there is a contradiction.

$$(16) \quad R_1 : h_2 h_3^{-2} h_2 h_3 = 1, \quad R_2 : h_2^2 h_3^{-1} (h_3^{-1} h_2)^2 h_3 = 1, \quad R_3 : h_2 h_3^{-1} (h_2^{-1} h_3)^3 h_2^{-1} h_3^{-1} h_2 = 1, \\ R_4 : h_2 (h_2 h_3^{-1})^4 h_2 h_3 = 1:$$

By interchanging h_2 and h_3 in (2) and with the same discussion, there is a contradiction.